

HAMILTONIAN PROPERTIES OF EARTHQUAKE FLOWS ON SURFACES WITH CLOSED GEODESIC BOUNDARY

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ABSTRACT. The Teichmüller space $\mathcal{T}_S(\mathbf{b})$ of hyperbolic metrics on a surface S with fixed lengths at the boundary components is symplectic. We prove that any sum of infinitesimal earthquakes on S that is tangent to $\mathcal{T}_S(\mathbf{b})$ is Hamiltonian, by providing a Hamiltonian \mathbb{L} . Such function extends the classical length map associated to a compactly supported measured geodesic lamination and shares with it some peculiar properties, such as properness and strict convexity along earthquakes paths under usual topological conditions. As an application, we prove that any non-Fuchsian affine representation of $\pi_1(S)$ into $\mathbb{R}^{2,1} \rtimes SO_0(2, 1)$ with cocompact discrete linear part is determined by the singularities of the two invariant regular domains in $\mathbb{R}^{2,1}$ pointed out by Barbot, once the boundary lengths are fixed.

INTRODUCTION

Let S be a surface of genus \mathbf{g} with \mathbf{n} closed mutually disjoint disks removed, with $\chi(S) = 2 - 2\mathbf{g} - \mathbf{n} < 0$. Consider the space \mathcal{T}_S of hyperbolic metrics on S whose completion \bar{S} has \mathbf{n} closed geodesic boundary components $\partial_1, \dots, \partial_{\mathbf{n}}$, up to diffeomorphisms of \bar{S} isotopic to the identity. Such metrics can be deformed via *left/right hyperbolic earthquakes*, which roughly speaking transform $h \in \mathcal{T}_S$ to $h' \in \mathcal{T}_S$ by shearing (S, h) towards the left/right along *measured geodesic laminations*, whose space is denoted by \mathcal{ML}_S . Weighted closed geodesics are the basic examples of elements of \mathcal{ML}_S . Thus, associated with each measured geodesic lamination λ there are the left and right earthquake maps $E_l^\lambda, E_r^\lambda: \mathcal{T}_S \rightarrow \mathcal{T}_S$.

Let us first consider when S is closed, i.e. $\mathbf{n} = 0$. The space of weighted closed geodesics is in this case dense in \mathcal{ML}_S . With every $\lambda \in \mathcal{ML}_S$ it is associated the *length map* $L_\lambda: \mathcal{T}_S \rightarrow \mathbb{R}$, defined for any ω -weighted closed geodesic c as $L_\lambda(h) = \omega \ell_h(c)$ and extended for $\lambda \in \mathcal{ML}_S$ by approximation. It was proved by Wolpert in [42] that E_l^λ is the Hamiltonian flow of $-L_\lambda$ with respect to the Weil-Petersson form ϖ_{WP} on $T\mathcal{T}_S$. The related Hamiltonian vector field is denoted by e_l^λ .

The aim of this paper is to extend such result when $\mathbf{n} > 0$. In such attempt, some tools and certain statements occurring in the closed case go missing. First of all, $\lambda \in \mathcal{ML}_S$ can contain geodesics *spiralling* near boundary components of S . This implies that λ can not be approximated by weighted closed geodesics, and a priori it is not clear how a length map L_λ can be defined. Moreover, \mathcal{T}_S is no longer a symplectic manifold (its dimension could even be odd). This can be bypassed by partitioning \mathcal{T}_S with submanifolds

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which are symplectic: for every $\mathbf{b} = (b_1, \dots, b_n) \in (\mathbb{R}_{>0})^n$, on the tangent of $\mathcal{T}_S(\mathbf{b})$, the space of metrics $h \in \mathcal{T}_S$ with fixed boundary lengths $b_i = \ell_h(\partial_i)$, a symplectic structure is induced by the one on $T\mathcal{T}_{2S}$, where $2S$ denotes the *double* of S . However, if $\lambda \in \mathcal{ML}_S$ has spiralling leaves then the *infinitesimal left earthquake* $e_l^\lambda \in \Gamma(T\mathcal{T}_S)$ is not tangent to $\mathcal{T}_S(\mathbf{b})$. There is a notion of signed intersection of a lamination λ near a boundary component ∂_i (see [16]). For any N -uple $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N) \in \mathcal{ML}_S^N$, the vector field $e_l^\boldsymbol{\lambda} = e_l^{\lambda_N} + \dots + e_l^{\lambda_1}$ is tangent to $\mathcal{T}_S(\mathbf{b})$ if and only if the sum of the signed intersections of $\lambda_1, \dots, \lambda_N$ near ∂_i is null for every i . We denote the space of such N -uples by $\mathcal{ML}_S^\#$. The main theorem can now be stated as follows.

Theorem A. *Given $\mathbf{b} \in (\mathbb{R}_{>0})^n$, the vector field $e_l^\lambda \in \Gamma(T\mathcal{T}_S(\mathbf{b}))$ is Hamiltonian for every $\lambda \in \mathcal{ML}_S^\#$.*

We provide a Hamiltonian $-\mathbb{L}_\lambda: \mathcal{T}_S(\mathbf{b}) \rightarrow \mathbb{R}$ which extends $-\sum L_{\lambda_n}$ to the case when $n > 0$. We also show that \mathbb{L}_λ is strictly convex (in a suitable sense) and proper if $\boldsymbol{\lambda} \in \mathcal{ML}_S^\#$ is a N -uple that *fills up* S , i.e. every simple closed non-trivial and non-peripheral curve meets the support of $\lambda_1 \cup \dots \cup \lambda_N$. We denote by $\mathcal{FML}_S^\#$ the space of filling couples in $\mathcal{ML}_S^\#$.

We provide an application within the study of flat Lorentzian structures, analogue to the compact case shown in [18]. Identifying $\mathbb{R}^{2,1}$ with the Lie algebra of $SL(2, \mathbb{R})$ (through the Killing form) and \mathcal{T}_S with the space \mathcal{R} of Fuchsian cocompact representations of $\pi_1(S)$, the tangent space $T\mathcal{T}_S$ can be identified with the space of affine deformations of elements of \mathcal{R} . Barbot showed in [3] that associated with $\rho = h + \tau \in \mathcal{R}$ there are two ρ -regular domains (as they are called in [8]) in $\mathbb{R}^{2,1}$. Each domain is determined by a lamination on the surface base point of ρ , viewed as the dual of the singularities of the domains (see [8]). The couple (λ_+, λ_-) of such laminations fills up S and satisfies the condition $\tau = e_l^{\lambda_-}(h) = -e_l^{\lambda_+}(h)$. We show that (λ_+, λ_-) determines ρ up to fixing the boundary lengths:

Theorem B. *The map $\Psi: T\mathcal{T}_S \rightarrow \mathcal{ML}_S^2$ associating $\rho = (h, \tau)$ with the couple (λ_+, λ_-) described above is a fibration over $\mathcal{FML}_S^\#$, the subset of $\mathcal{ML}_S^\#$ of filling couples. The fiber is isomorphic to \mathbb{R}^n .*

This paper is organized as follows. In the first part of Section 1 we recall general notions about measured geodesic laminations and hyperbolic earthquakes on S . After that, we proceed to give to \mathcal{ML}_S a manifold structure compatible with the weak*-convergence topology and we study smoothness of infinitesimal earthquakes. Finally, we endow $\mathcal{T}_S(\mathbf{b})$ with a symplectic structure.

Section 2 is devoted to the construction of \mathbb{L}_λ , starting from the Hamiltonian condition and decomposing any $\boldsymbol{\lambda} \in \mathcal{ML}_S^\#$ in the union of simple couples in $\mathcal{ML}_S^\#$, in a suitable sense. After defining \mathbb{L} for these simple couples and checking the Hamiltonian condition, we provide \mathbb{L}_λ for generic $\boldsymbol{\lambda} \in \mathcal{ML}_S^\#$. Properness and strict convexity of \mathbb{L}_λ are proved in Section 3, where is also computed $\text{Hess } \mathbb{L}_\lambda$ at its critical point.

In Section 4 we apply such results to the study of $\Psi: T\mathcal{T}_S \rightarrow \mathcal{FML}_S^\#$.

1. EARTHQUAKES AND MEASURED GEODESIC LAMINATIONS

Given a topological surface S obtained by removing \mathbf{n} closed mutually disjoint disks from a compact surface of genus \mathbf{g} with Euler characteristic $\chi(S) = 2 - 2\mathbf{g} - \mathbf{n} < 0$, let

$$\mathcal{T}_S = \{\text{hyperbolic metrics on } S \text{ whose completion } \bar{S} \text{ has } \mathbf{n} \text{ closed geodesic boundary components}\} / \text{Diff}_0(\bar{S}),$$

where $\text{Diff}_0(\bar{S})$ denotes the group of the diffeomorphisms of \bar{S} isotopic to the identity. We will refer to the boundary components of S as $\partial_1, \dots, \partial_{\mathbf{n}}$.

1.1. Measured geodesic laminations.

Definition 1.1. Given a hyperbolic metric h on S , a geodesic lamination on (S, h) is the data λ of a family of mutually disjoint complete simple geodesics (called the leaves of λ) whose union is a closed subset (called the support of λ and denoted by $\text{supp}(\lambda)$) of S . A measured geodesic lamination of S is the data of a geodesic lamination λ and a transverse measure meas_λ , that is a measure defined on the arcs on S transverse to each leaf of λ and with endpoints in $S \setminus \text{supp}(\lambda)$ such that

- $\text{meas}_\lambda(c) \neq 0$ if and only if $c \cap \text{supp}(\lambda) \neq \emptyset$;
- if there exists an isotopy between two arcs c_1 and c_2 realized through arcs transverse to λ then $\text{meas}_\lambda(c_1) = \text{meas}_\lambda(c_2)$.

Weighted multicurves are the simplest examples of measured geodesic lamination on S . The support is the finite union of simple closed mutually disjoint non trivial geodesics γ_i . Chosen real positive numbers ω_i (called weights) respectively assigned to γ_i , the transverse measure is given by

$$c \mapsto \sum_i \omega_i \cdot \#(\gamma_i \cap c)$$

for any arc c transverse to $\bigcup \gamma_i$.

It is known (see [20]) that the Lebesgue measure of the support of a geodesic lamination is zero.

If $h \in \mathcal{T}_S$ then any measured geodesic lamination λ on (S, h) has a maximal compact sublamination $\lambda^{(0)}$, in the sense that if μ is a sublamination of λ with compact support in S then μ is a sublamination of $\lambda^{(0)}$ too. Each leaf of $\text{supp}(\lambda) \setminus \text{supp}(\lambda^{(0)})$ is homeomorphic to \mathbb{R} and spirals near two boundary components (possibly coincident) of S (see Figure 1).

If we denote by $\mathcal{ML}_{(S, h)}$ the measured geodesic laminations on (S, h) with $h \in \mathcal{T}_S$, being a space of measures it seems natural to provide it with the topology of the weak-convergence of measures (sometimes also called weak*-convergence). It is known (see Section 1.7 of [34]) that for every h_1, h_2

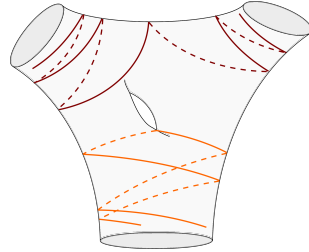


FIGURE 1. A geodesic lamination with two spiralling leaves

in \mathcal{T}_S there is a homeomorphism $F : \mathcal{ML}_{(S, h_1)} \rightarrow \mathcal{ML}_{(S, h_2)}$ so that, roughly speaking, $\text{supp}(F(\lambda))$ is obtained straightening with respect to h_2 the leaves of $\text{supp}(\lambda)$. This suggests that it makes sense to associate \mathcal{T}_S with the space \mathcal{ML}_S of measured laminations, whose support is only a topological data; this space inherits the weak convergence topology. Finally, define

$$\mathcal{ML}_S^c = \{\lambda \in \mathcal{ML}_S \mid \lambda = \lambda^{(0)}\},$$

the subspace of laminations with compact support. The following theorem is a well known result (see [34]).

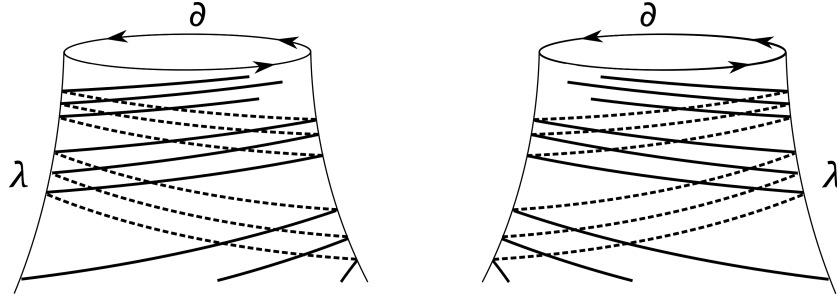


FIGURE 2. Respectively, positive and negative sense of spiralling

Theorem 1.1. *The space of weighted multicurves on S is dense in \mathcal{ML}_S^c .*

Let us fix for a moment $h \in \mathcal{T}_S$ and consider a measured geodesic lamination λ on (S, h) . If a leaf of λ is not contained in a compact subset of S , then, in order to be a complete geodesic with no self-intersections, it must spiral along one or two connected components of ∂S . There are two possible senses of spiralization, as shown in Figure 2.

In particular, if a leaf l of λ spirals near ∂ , then for every lift $\tilde{\partial} \subset \mathbb{H}^2$ of ∂ there is an ε -neighbourhood of $\tilde{\partial}$ where the preimage of l is the $\text{Stab}(\tilde{\partial})$ -orbit of any lift \tilde{l} of l sharing an ideal endpoint of $\tilde{\partial}$, as in Figure 3. See also Lemma 2.3.

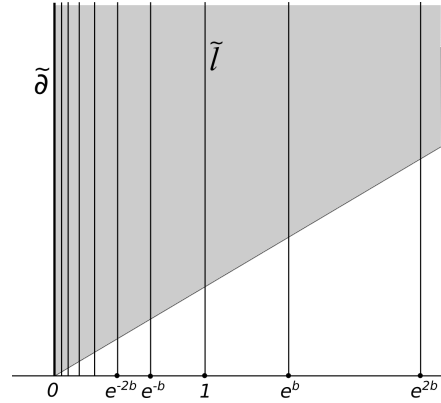


FIGURE 3

It is possible to define the *mass* $\iota(\partial, \lambda)$ of ∂ with respect to λ , a positive number that encodes how much the measure of λ is concentrated near ∂ . It is constructed as follows. For every $x \in N_\varepsilon(\partial)$ denote by c_x the loop with vertex at x parallel at ∂ such that $c_x \setminus \{x\}$ is an open geodesic arc. Since $\text{meas}_\lambda(c_x) = \text{meas}_\lambda(c_y)$ for every $x, y \in N_\varepsilon$, as shown in [18], Subsection 2.3, it is well defined the mass $\iota(\partial, \lambda) = \text{meas}_\lambda(c_x)$. Moreover, $\iota(\partial, \lambda) = 0$ if and only if $\text{supp}(\lambda) \cap N_\varepsilon = \emptyset$. The mass of ∂ does not take in account in which sense λ spirals. Fix once for all an orientation of ∂S . Such choice defines a

positive and a negative sense of spiralization around ∂ , as in Figure 2. It is now possible to define the *signed mass* $m(\partial, \lambda)$ of ∂ with respect to λ as

$$(1) \quad m(\partial, \lambda) = \begin{cases} +\iota(\partial, \lambda) & \text{if } \lambda \text{ spirals in the positive sense around } \partial \\ -\iota(\partial, \lambda) & \text{if } \lambda \text{ spirals in the negative sense around } \partial \end{cases}.$$

Remark 1.1. The signed mass of ∂ with respect to λ is positive (respectively negative) if and only if for every orientated lift of ∂ on \mathcal{H} its ending (respectively starting) ideal endpoint is contained in the set of the ideal points of the whole preimage of λ .

1.2. Hyperbolic earthquakes. Let \mathcal{H} be a convex subset of \mathbb{H}^2 with geodesic boundary.

Definition 1.2. Given a geodesic lamination λ in \mathcal{H} , a left (respectively right) hyperbolic earthquake on \mathcal{H} along λ is an injective (possibly discontinuous) map $\tilde{E} : \mathcal{H} \rightarrow \mathbb{H}^2$ such that

- the restriction of \tilde{E} on a stratum of λ is an isometry;
- denoting by $A_F \in PSL(2, \mathbb{R})$ the isometry of \mathbb{H}^2 extending $\tilde{E}|_F$ for every stratum F , the *comparison map*

$$\text{cmp}(F, G) = A_F^{-1} \circ A_G : \mathbb{H}^2 \rightarrow \mathbb{H}^2$$

between two different strata F and G of λ is a hyperbolic transformation whose axis weakly separates F and G and which translates to the left (respectively right), as viewed from F .

The lamination λ is called *fault locus* of the earthquake \tilde{E} .

It turns out that $\tilde{E}(\mathcal{H})$ is still a convex subset of \mathbb{H}^2 with geodesic boundary, as a consequence of Lemma 8.4 in [16].

Given a surface S and two hyperbolic metrics h_1, h_2 on S , set $S_i = (S, h_i)$ for $i = 1, 2$. Suppose that the universal covering $\mathcal{H}_i \subset \mathbb{H}^2$ of S_i is convex with geodesic boundary. A bijective map $E : S_1 \rightarrow S_2$ is a left (respectively right) hyperbolic earthquake if it has a lifting $\tilde{E} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ which is a left (respectively right) hyperbolic earthquake on \mathcal{H}_1 .

The fault locus can be endowed with a transverse measure encoding the shearing of the earthquake, obtaining a measured geodesic lamination: the ω -weighted curve c . This can be done in general, as stated in the following ([38], Proposition 6.1).

Proposition 1.2. *A measured geodesic lamination $\lambda \subset \mathcal{H}$ is associated to any earthquake so that $\text{supp}(\lambda)$ coincides with the fault locus; if $a : [0, 1] \rightarrow \mathcal{H}$ is an arc with endpoints in $\mathcal{H} \setminus \lambda$ then*

$$\text{meas}_\lambda(a) = \inf_{P \text{ partition of } [0,1]} \sum_{i=1}^{I_P} T(\text{cmp}(A_{F_{i-1}}, A_{F_i}))$$

where for every partition $P = (0 = t_0, t_1, t_2, \dots, t_{I_P} = 1)$ of $[0, 1]$ the stratum F_i of λ is the one containing t_i . Here $T(B)$ denotes the translation length of a hyperbolic transformation B . \square

Moreover, Thurston showed that different earthquakes produce different measured geodesic laminations (see [38]). The converse holds, since we did not suppose that \tilde{E} is surjective. See [16] for further details.

There is a natural non surjective immersion of \mathcal{T}_S into the Teichmüller space $\overline{\mathcal{T}}_S$ of hyperbolic metrics on S of finite area whose completion has compact geodesic boundary. A metric in $\overline{\mathcal{T}}_S$ can have cusps at some punctures of S . Associated with $\lambda \in \mathcal{ML}_S$, there are a left and a right earthquake map between \mathcal{T}_S and $\overline{\mathcal{T}}_S$:

$$E_l^\lambda, E_r^\lambda: \mathcal{T}_S \rightarrow \overline{\mathcal{T}}_S.$$

Proposition 3.3 in [16] shows explicitly how right and left earthquakes change the length of the boundary components $\partial_1, \dots, \partial_n$ of S : for every $h \in \mathcal{T}_S$ and $\lambda, \mu \in \mathcal{ML}_S$

$$(2) \quad \begin{cases} \ell_{E_l^\lambda(h)}(\partial_i) = |\ell_h(\partial_i) - m(\partial_i, \lambda)| \\ \ell_{E_r^\mu(h)}(\partial_i) = |\ell_h(\partial_i) + m(\partial_i, \mu)| \end{cases}.$$

Fix $\mathbf{b} = (b_1, \dots, b_n) \in (\mathbb{R}_{>0})^n$ and set

$$\mathcal{T}_S(\mathbf{b}) = \{h \in \mathcal{T}_S \mid \ell_h(\partial_i) = b_i \quad \forall i = 1, \dots, n\}.$$

Clearly,

$$\mathcal{T}_S = \bigcup_{\mathbf{b} \in (\mathbb{R}_{>0})^n} \mathcal{T}_S(\mathbf{b}).$$

In this paper we are interested in N -uples $\boldsymbol{\lambda} \in \mathcal{ML}_S^N$ for which the vector field

$$e_t^\lambda(h) = \frac{d}{dt}\bigg|_0 \left(E_l^{t\lambda_N} \circ \dots \circ E_l^{t\lambda_1}(h) \right)$$

is tangent to $\mathcal{T}_S(\mathbf{b})$, with $\mathbf{b} \in (\mathbb{R}_{>0})^n$. Now, for every $i = 1, \dots, n$, if $h_t = E_l^{t\lambda_N} \circ \dots \circ E_l^{t\lambda_1}(h)$ then, using (2) for t sufficiently small,

$$\ell_{h_t}(\partial_i) = \ell_h(\partial_i) - tm(\partial_i, \lambda_1) - tm(\partial_i, \lambda_2) - \dots - tm(\partial_i, \lambda_N)$$

and so $e_t^\lambda \in \Gamma(T\mathcal{T}_S(\mathbf{b}))$ if and only if

$$0 = \frac{d}{dt}\bigg|_0 \ell_{h_t}(\partial_i) = -m(\partial_i, \lambda_1) - m(\partial_i, \lambda_2) - \dots - m(\partial_i, \lambda_N)$$

for every $h \in \mathcal{T}_S(\mathbf{b})$ and $i = 1, \dots, n$. Notice that such condition is actually independent on \mathbf{b} . Thus, we introduce the space

$$\mathcal{ML}_S^\# = \left\{ \boldsymbol{\lambda} \in \mathcal{ML}_S^N \mid \sum_{n=1}^N m(\partial_i, \lambda_n) = 0 \quad \forall i = 1, \dots, n \right\}.$$

Remark 1.2. When $N = 1$ then $\mathcal{ML}_S^\# = \mathcal{ML}_S^c$. Since classical results are already known for compactly supported laminations, we will suppose from now on that $N \geq 2$.

1.3. The topology of \mathcal{ML}_S . Now we are going to give to \mathcal{ML}_S a manifold structure. First let us introduce the straightening ν^R of a measured lamination $\nu \in \mathcal{ML}_S$. If γ is a spiralling geodesic between two connected components ∂_i and ∂_j of ∂S , consider its preimage Γ on the universal cover $\mathcal{H} \subset \mathbb{H}^2$. Every connected component of Γ is a geodesic $\tilde{\gamma}$ with endpoints in the (ideal closure) of certain lifts $\tilde{\partial}_i$ and $\tilde{\partial}_j$ of ∂_i and ∂_j respectively. If we replace each $\tilde{\gamma}$ with the geodesic arc $\tilde{\gamma}^R$ with endpoints on $\tilde{\partial}_i$ and $\tilde{\partial}_j$ perpendicular to $\tilde{\partial}_i$ and $\tilde{\partial}_j$ and we project $\tilde{\gamma}^R$ on S , we obtain a geodesic arc γ^R on S normal to ∂_i and ∂_j with endpoints on ∂_i and ∂_j . For each $\nu \in \mathcal{ML}_S$ denote by ν^R the set of geodesic (weighted) arcs obtained by ν replacing each spiralling geodesic γ of ν with γ^R .

Consider the set $\mathcal{ML}_S^R = \{\nu^R \mid \nu \in \mathcal{ML}_S\}$. This space is a submanifold of the space of measured laminations (that we denote by \mathcal{ML}_S^\dagger) studied in [1]; we will mention only the necessary details. Using the notation of [1], we fix a pant decomposition

$$P = \{C_1, \dots, C_{3g-3+n}, B_1 = \partial_1, \dots, B_n = \partial_n\}$$

of S with internal curves C_1, \dots, C_{3g-3+n} and boundary curves $B_1 = \partial_1, \dots, B_n = \partial_n$. Every lamination $\sigma \in \mathcal{ML}_S^\dagger$ has coordinates

$$(DT(\sigma, C_1), \dots, DT(\sigma, C_{3g-3+n}), \hat{\theta}(\sigma, B_1), \dots, \hat{\theta}(\sigma, B_n))$$

where $DT(\sigma, C_i) \in \mathbb{R}^2$ depends on the behaviour of σ in a regular neighbourhood of C_i and $\hat{\theta}(\sigma, \partial_i) \in \mathbb{R}$ depends on the behaviour with respect to the boundary component ∂_i . Following their constructions, it turns out that, for every $\nu \in \mathcal{ML}_S$, $\hat{\theta}(\nu^R, \partial_i) = \iota(\nu, \partial_i) \geq 0$. So if we consider the coordinates $\Theta_P: \mathcal{ML} \rightarrow \mathbb{R}^{6g-6+3n}$ such that

$$(3) \quad \Theta_P(\nu) = (DT(\nu^R, C_1), \dots, DT(\nu^R, C_{3g-3+n}), m(\nu, \partial_1), \dots, m(\nu, \partial_n))$$

for $\nu \in \mathcal{ML}_S$, where $m(\nu, \partial_i)$ is the signed mass defined by (1), we provide \mathcal{ML}_S with a manifold structure. Such coordinates depend on the pant decomposition P ; however, if P' is another pant decomposition, notice that the last n coordinates does not depend on the pant decomposition, whereas applying the results in [1] the change of coordinates of the other components is smooth.

Even if the projection $\mathcal{ML}_S \rightarrow \mathcal{ML}_S^R$ is not injective, the map Θ_P is injective, since we have avoided the ambiguity given by the spiralling senses around ∂S .

It is shown in [1] that the topology on \mathcal{ML}_S^\dagger coincides with the topology of the weak*-convergence of measures. We are interested to show that also for \mathcal{ML}_S the topology is the one of weak*-convergence of measures.

Lemma 1.3. *Consider a sequence λ_n converging to λ in the manifold \mathcal{ML}_S . If $\lambda^{[s]}$ is the sublamination of λ made by spiralling leaves, then the support of $\lambda^{[s]}$ is contained in λ_n for n sufficiently big. In particular, there exist decompositions*

$$\begin{aligned} \lambda_n &= \lambda_n^{[c]} \cup \lambda_n^{[s]} \cup \lambda_n^{[v]}, \\ \lambda &= \lambda^{[cc]} \cup \lambda^{[s]} \cup \lambda^{[cv]} \end{aligned}$$

such that, up to passing to a subsequence,

- $\lambda_n^{[c]}$ is the maximal compact sublamination of λ_n , and $\lambda_n^{[c]}$ converges to $\lambda^{[cc]}$;
- $\lambda^{[s]}$ is the sublamination of λ whose support consists of the spiralling leaves of λ , and $\lambda_n^{[s]}$ is the maximal sublamination of λ_n such that $\text{supp}(\lambda_n^{[s]}) = \text{supp}(\lambda^{[s]})$; moreover, $\lambda_n^{[s]}$ tends to $\lambda^{[s]}$;
- $\lambda_n^{[v]}$ is the complementary of $\lambda_n^{[s]}$ in the spiralling part of λ_n , so that $\lambda_n^{[v]}$ converges to the compact lamination $\lambda^{[cv]}$.

Proof. We prove that if l_n is a sequence of leaves of λ_n converging to a leaf $l \in \lambda^{[s]}$, then $l_n = l$ for n big. The claim directly implies the statement. Let us prove the claim.

Consider a leaf l of $\lambda^{[s]}$, going say between the boundary components ∂ and ∂' of S . On the universal covering $\mathcal{H} \subset \mathbb{H}^2$ of S , consider a lift \tilde{l} of l , going from $\tilde{\partial}$ and $\tilde{\partial}'$, the boundary components of $\partial\mathcal{H}$ who projects onto ∂ and ∂' respectively. The straightening \tilde{l}^R of \tilde{l} has an endpoint $z \in \tilde{\partial}$. There is a δ -neighbourhood U of \tilde{l}^R in \mathcal{H} such that for every $u \in (\bar{U} \cap \tilde{\partial}) \setminus \{z\}$ the complete geodesic of \mathbb{H}^2 normal to $\tilde{\partial}$ passing through u must intersect $\tilde{\partial}'$, but this intersection cannot be orthogonal, so if a lamination $\nu \in \mathcal{ML}_S^\dagger$ meets $U \cap \tilde{\partial}$, then it must contain the leaf l . Thus, leaves of $(\lambda^{[s]})^R$ must be contained in $(\lambda_n \setminus \lambda_n^{[c]})^R$ for big n , and in fact $(\lambda^{[s]})^R$ must be the limit of the sublamination $(\lambda_n^{[ss]})^R$ made by the leaves of $(\lambda_n \setminus \lambda_n^{[c]})^R$ whose weight is not tending to zero. \square

Proposition 1.4. *If $\lambda_n \rightarrow \lambda$ in \mathcal{ML}_S then for every arc α on S with endpoints in $S \setminus (\text{supp}(\lambda) \cup \bigcup \text{supp}(\lambda_n))$ and for every $\varphi \in C_c^\infty(\alpha)$*

$$\int_\alpha \varphi d(\text{meas}_{\lambda_n}) \xrightarrow{n \rightarrow \infty} \int_\alpha \varphi d(\text{meas}_\lambda).$$

Proof. From now on, for simplicity we will write $d\lambda_n$ and $d\lambda$ respectively for $d(\text{meas}_{\lambda_n})$ and $d(\text{meas}_\lambda)$.

Take the decomposition

$$\begin{aligned} \lambda_n &= \lambda_n^{[c]} \cup \lambda_n^{[s]} \cup \lambda_n^{[v]}, \\ \lambda &= \lambda^{[cc]} \cup \lambda^{[s]} \cup \lambda^{[cv]} \end{aligned}$$

provided by Lemma 1.3, and consider the induced decomposition on the double straightenings Λ_n, Λ of λ_n, λ respectively:

$$\begin{aligned} \Lambda_n &= \Lambda_n^{[c]} \cup \Lambda_n^{[s]} \cup \Lambda_n^{[v]}, \\ \Lambda &= \Lambda^{[cc]} \cup \Lambda^{[s]} \cup \Lambda^{[cv]}. \end{aligned}$$

Notice that the weights of the leaves of $\Lambda_n^{[v]}$ are going to 0, since the masses of $\Lambda_n^{[v]}$ at the boundary of S are vanishing.

Fixed $\epsilon > 0$ and denoting by

$$\begin{aligned} T_1 &= \left| \int_{\alpha} \varphi d\lambda_n^{[c]} - \int_{\alpha} \varphi d\lambda^{[cc]} \right| \\ T_2 &= \left| \int_{\alpha} \varphi d\lambda_n^{[s]} - \int_{\alpha} \varphi d\lambda^{[s]} \right| \\ T_3 &= \left| \int_{\alpha} \varphi d\lambda_n^{[v]} - \int_{\alpha} \varphi d\lambda^{[cv]} \right| \end{aligned}$$

it suffices to show that for n sufficiently large $T_1 + T_2 + T_3 \leq 6\epsilon$.

It is easy to estimate $T_1 \leq \epsilon$ and $T_2 \leq \epsilon$ for n large enough, due respectively to the compact and discrete nature of the involved sublaminations. The term T_3 requires more attention. First of all, let us split it as

$$\begin{aligned} T_3 &\leq \left| \int_{\alpha} \varphi d\lambda_n^{[v]} - \int_{\alpha} \varphi d\Lambda_n^{[v]} \right| + \left| \int_{\alpha} \varphi d\Lambda_n^{[v]} - \int_{\alpha} \varphi d\lambda^{[cv]} \right| = \\ &= \left| \int_{\alpha} \varphi d\lambda_n^{[v]} - \int_{\alpha} \varphi d\Lambda_n^{[v]} \right| + \left| \int_{\alpha} \varphi d\Lambda_n^{[v]} - \int_{\alpha} \varphi d\Lambda^{[cv]} \right|. \end{aligned}$$

The second term of the last member is not greater than ϵ for n large enough, since $\Lambda_n^{[v]} \rightarrow \Lambda^{[cv]}$. Let us consider the first one. Fix a lift $\tilde{\alpha}$ of α in the universal covering of S . For every leaf $\tilde{\delta}$ of the preimage of a leaf δ of $\Lambda_n^{[v]}$ denote by $D_{\tilde{\alpha}}(\tilde{\delta})$ the minimum between the lengths of the two connected components of $\tilde{\delta}^R \setminus \tilde{\alpha}$ if $\tilde{\delta}^R \cap \tilde{\alpha}$ is non empty. See also Figure 4. There is a constant $M = M(\alpha, \epsilon) > 0$ such that if $D_{\tilde{\alpha}}(\tilde{\delta}) > M$ then the ideal endpoints

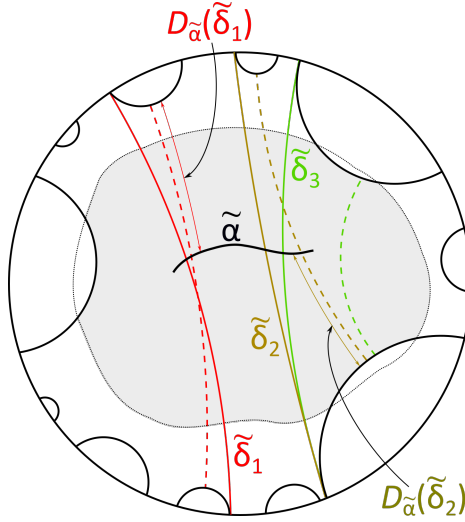


FIGURE 4. The points in the grey region have distance from $\tilde{\alpha}$ less than $M(\alpha, \epsilon)$; the leaf $\tilde{\delta}_1$ of $\tilde{\lambda}_n^{[v]}$ is contained in $\tilde{\lambda}_n^{[v]+}$, while $\tilde{\delta}_2$ and $\tilde{\delta}_3$ are contained in $\tilde{\lambda}_n^{[v]-}$

of $\tilde{\delta}$ are close to the ones of the prolongation of $\tilde{\delta}^R$, in the Euclidean sense, so that

$$\left| \int_{\alpha} \varphi d\lambda_n^{[v]+} - \int_{\alpha} \varphi d\Lambda_n^{[v]+} \right| \leq \epsilon$$

for n sufficiently large, where $\lambda_n^{[v]+}$ is the sublamination of $\lambda_n^{[v]}$ of the leaves δ whose straightening meets α having $D_{\tilde{\alpha}}(\tilde{\delta}) > M$, while $\Lambda_n^{[v]+}$ is the doubled straightening of $\lambda_n^{[v]+}$. Set $\lambda_n^{[v]-} = \lambda_n^{[v]} \setminus \lambda_n^{[v]+}$ and $\Lambda_n^{[v]-} = \Lambda_n^{[v]} \setminus \Lambda_n^{[v]+}$. Now

$$\begin{aligned} & \left| \int_{\alpha} \varphi d\lambda_n^{[v]} - \int_{\alpha} \varphi d\Lambda_n^{[v]} \right| \leq \left| \int_{\alpha} \varphi d\lambda_n^{[v]+} - \int_{\alpha} \varphi d\Lambda_n^{[v]+} \right| + \\ & + \left| \int_{\alpha} \varphi d\lambda_n^{[v]-} - \int_{\alpha} \varphi d\Lambda_n^{[v]-} \right| \leq \epsilon + \left| \int_{\alpha} \varphi d\lambda_n^{[v]-} \right| + \left| \int_{\alpha} \varphi d\Lambda_n^{[v]-} \right|. \end{aligned}$$

Actually, $\Lambda_n^{[v]-}$ (and consequently $\lambda_n^{[v]-}$) is vanishing, since its number of leaves is bounded from above by a constant depending only on the geometry of S : on its universal covering \mathcal{H} , it is easy to see that the number of connected components of $\partial\mathcal{H}$ distant at most M from $\tilde{\alpha}$, which has compact support, are finite. Moreover, the weights of the leaves of $\Lambda_n^{[v]}$ are going to 0, as $\lambda_n^{[v]}$ converges to a compact lamination. Thus, for n big,

$$\left| \int_{\alpha} \varphi d\lambda_n^{[v]-} \right| + \left| \int_{\alpha} \varphi d\Lambda_n^{[v]-} \right| \leq 2\epsilon.$$

□

1.4. Infinitesimal earthquakes. Associated with $\lambda \in \mathcal{ML}_S$, there is the vector field

$$\begin{aligned} e_l^\lambda: \mathcal{T}_S &\rightarrow T\mathcal{T}_S \\ h &\mapsto \frac{d}{dt}\bigg|_0 (E_l^{t\lambda}(h)) \end{aligned}$$

called the *infinitesimal left earthquake* along λ .

Proposition 1.5. *For every $\lambda \in \mathcal{ML}_S$, the vector field e_l^λ is a smooth vector field on \mathcal{T}_S .*

Proof. Let us suppose λ has a non empty compact sublamination. Decompose $\lambda = \lambda_c \oplus \lambda_s$ as the sum of the compact maximal sublamination with the spiralling sublamination. Then e_l^λ can be decomposed as $e_l^{\lambda_c} + e_l^{\lambda_s}$. By classical results, $e_l^{\lambda_c}$ is smooth. So we can suppose $\lambda = \lambda_s$ and consider only this case.

It is convenient to see $\mathcal{T}_S(\mathbf{b})$ as the space of faithful discrete representations $h: \pi_1(S) \rightarrow PSL(2, \mathbb{R})$ with conditions that fix the images of peripheral loops, up to conjugacy. For every $h \in \mathcal{T}_S(\mathbf{b})$, consider the universal covering \mathcal{H} of S such that $h(\pi_1(S)) \backslash \mathcal{H} \cong S$ and fix a point $z \in \mathcal{H}$; the infinitesimal earthquake regarded as an element of the cohomology $H^1(\pi_1(S), \mathbb{R}^{2,1})$ is represented (see [32], [2], [18]) by the element $e_l^\lambda(h): \pi_1(S) \rightarrow \mathfrak{so}(2, 1) \cong \mathbb{R}^{1,2}$ has the form

$$\gamma \mapsto \int_{\mathcal{G}} v(r) \chi_{\mathcal{G}(\gamma)}(r) d\lambda$$

where

- the space $\mathfrak{so}(2, 1)$ is the Lie algebra of $SO(2, 1)$,
- the space

$$\mathcal{G} \cong (S^1 \times S^1) \setminus \text{diag}(S^1)$$

is the set of oriented geodesics on \mathbb{H}^2 ,

- the map

$$v: \mathcal{G} \rightarrow \mathfrak{so}(2, 1)$$

sends $r \in \mathcal{G}$ to the infinitesimal generator of the hyperbolic transformations on the hyperboloid $\mathbb{H}^2 \subset \mathbb{R}^{2,1}$ with r as oriented axis,

- the set $\mathcal{G}(\gamma) \subset \mathcal{G}$ is the subset containing the leaves of $\text{supp}(\lambda)$, oriented consistently with the λ -earthquake whose lifting $\tilde{\lambda}$ on \mathcal{H} fixes z , that meet the geodesic arc $[z, h(\gamma)(z)]$,
- $d\lambda$ denotes $d\text{meas}_\lambda$.

Given a smooth family $(h_t)_{t \in I} \subset \mathcal{T}_S(\mathbf{b})$, where I is an interval of \mathbb{R} containing 0, we want to show that for every $\gamma \in \pi_1(S)$ the map $t \mapsto e_t^\lambda(h_t)(\gamma)$ is smooth. Consider the relative covers \mathcal{H}_t and subsets $\mathcal{G}_t(\gamma) \subset \mathcal{G}$. Denote by $\tilde{\lambda}_t$ the realization of $\tilde{\lambda}$ in \mathcal{H}_t . Now

$$e_t^\lambda(h_t)(\gamma) = \int_{\mathcal{G}} v(r) \chi_{\mathcal{G}_t(\gamma)}(r) d\lambda_t.$$

For every $t \in I$ there exists a homeomorphism $\zeta_t: \partial\mathcal{H}_0 \rightarrow \partial\mathcal{H}_t$ which is h_t -equivariant, i.e.

$$\zeta_t(h_t(\beta)(x)) = h_t(\beta)(\zeta_t(x)) \quad \forall x \in \partial\mathcal{H}_0 \quad \forall \beta \in \pi_1(S),$$

and such that for every x that is an endpoint of an axis of $h_0(\alpha)$ for some $\alpha \in \pi_1(S)$ the map $t \mapsto \zeta_t(x)$ is smooth. It induces a map

$$Z_t = (\zeta_t)_*: \mathcal{G} \xrightarrow{\sim} \mathcal{G}.$$

It turns out that $\lambda_t = Z_t(\lambda_0)$, in the obvious sense. Notice that the endpoints of the leaves of λ_t are also endpoints of boundary components for every $t \in I$. Also, $\mathcal{G}_t(\gamma)(Z_t(s)) = \mathcal{G}_0(\gamma)(s)$ for every $s \in \mathcal{G}$. Now we have

$$e_t^\lambda(h_t)(\gamma) = \int_{\mathcal{G}} v(r) \chi_{\mathcal{G}_t(\gamma)}(r) dZ_t(\lambda_0) = \int_{\mathcal{G}} v(Z_t(s)) \chi_{\mathcal{G}_0(\gamma)}(s) d\lambda_0.$$

The integrand of the latter member is a smooth function of t , so we get that $t \mapsto e_t^\lambda(h_t)(\gamma)$ is smooth for every $\gamma \in \pi_1(S)$. \square

Remark 1.3. From the proof of the previous proposition we also get that if λ_n is a sequence of laminations converging to λ in \mathcal{ML}_S then $e_t^{\lambda_n}$ converges to e_t^λ in $\Gamma(T\mathcal{T}_S)$ with the C^∞ topology.

2. LENGTH MAP

This section is devoted to find a Hamiltonian $-\mathbb{L} = -\mathbb{L}_\lambda$, given any $\lambda \in \mathcal{ML}_S^\#$, for the vector field $e_t^\lambda = e_t^{\lambda_1} + \dots + e_t^{\lambda_N}$ with respect to a symplectic form on $T\mathcal{T}_S(\mathbf{b})$, provided in the first subsection. After an heuristic computation of $d\mathbb{L}$ (Subsect. 2.2), we decompose λ in simpler couples still lying in $\mathcal{ML}_S^\#$ such that the sum of their infinitesimal earthquakes gives e_t^λ (Subsect. 2.3). For such couples we define \mathbb{L} (Subsect. 2.4) and show that $d\mathbb{L}$ is what we expect (Subsect. 2.5). Finally, \mathbb{L}_λ will be constructed as the sum of such length maps (Subsect. 2.6).

2.1. The symplectic structure of $\mathcal{T}_S(\mathbf{b})$. Fix $\mathbf{b} = (b_1, \dots, b_n) \in (\mathbb{R}_{>0})^n$ once for all and consider

$$\mathcal{T}_S(\mathbf{b}) = \{h \in \mathcal{T}_S \mid \ell_h(\partial_k) = b_k \ \forall k = 1, \dots, n\}.$$

A pant decomposition of S with (internal) curves κ_i induces the coordinates

$$(\mathbf{l}, \boldsymbol{\tau}, \boldsymbol{\beta}) = (l_1, \dots, l_{3g-3+n}, \tau_1, \dots, \tau_{3g-3+n}, \beta_1, \dots, \beta_n)$$

on \mathcal{T}_S , where l_j denotes the length of κ_j , τ_j the twist factor of κ_j , and β_i the length of the boundary component ∂_i of S . The space $\mathcal{T}_S(\mathbf{b})$ is the submanifold of \mathcal{T}_S individuated by the n equations $\boldsymbol{\beta} = \mathbf{b}$.

If μ has not compact support then there exists $i \in \{1, \dots, n\}$ such that $m_i = m(\partial_i, \mu) \neq 0$, so we have

$$\ell_{E_t^\mu(h)}(\partial_i) = |b_i - tm_i| \neq b_i$$

for $t \in (0, \varepsilon)$ with ε sufficiently small; such a linear behaviour shows that if $h \in \mathcal{T}_S(\mathbf{b})$ then $e_t^\mu(h)$ does not lie in $T_h \mathcal{T}_S(\mathbf{b})$. However, for every $\boldsymbol{\lambda}$ in

$$\mathcal{ML}_S^\# = \left\{ \boldsymbol{\mu} = (\mu_1, \dots, \mu_N) \in \mathcal{ML}_S^N : \sum_{n=1}^N m(\partial_i, \mu_n) = 0, \ i = 1, \dots, n \right\}$$

e_t^λ is a tangent vector field of $\mathcal{T}_S(\mathbf{b})$, as shown at the end of Subsection 1.2. We need to provide $\mathcal{T}_S(\mathbf{b})$ with a symplectic form ϖ . However, there is a natural Weil-Petersson form on $\mathcal{T}_S(\mathbf{b})$ obtained in the following way. Let $2S$ be the double of S along its boundary. Choose a pant decomposition $\kappa_1^\pm, \dots, \kappa_{6(g-1)+2n}^\pm, \partial_1, \dots, \partial_n$ on $2S$ invariant by the natural involution. Let ϖ_{WP} denote the Weil-Petersson form on the Teichmüller space \mathcal{T}_{2S} of $2S$. It can be written as

$$\varpi_{\text{WP}} = \sum_{j=1}^{6(g-1)+2n} (d\ell_j^+ \wedge d\tau_j^+ + d\ell_j^- \wedge d\tau_j^-) + \sum_{i=1}^n d\ell_i^0 \wedge d\tau_i^0$$

where ℓ_j^\pm and τ_j^\pm denote respectively the length coordinate and the twist coordinate relative to κ_j^\pm in $2S$, while ℓ_i^0 and τ_i^0 denote respectively the length and twist coordinate relative to ∂_i . Consider the natural immersion $f: \mathcal{T}_S(\mathbf{b}) \rightarrow \mathcal{T}_{2S}$ that doubles a metric on S . With the 2-form

$$\varpi = f^* \varpi_{\text{WP}} = 2 \sum_{j=1}^{6(g-1)+2n} d\ell_j \wedge d\tau_j,$$

where ℓ_j and τ_j denote respectively the length coordinate and the twist coordinate relative to $f^*(\kappa_j^+) = f^*(\kappa_j^-)$, it turns out that $(\mathcal{T}_S(\mathbf{b}), \varpi)$ is a symplectic manifold.

2.2. Hamiltonian conditions. Consider a simple closed curve γ not isotopic to a boundary component. Choose a pant decomposition $\{\gamma, \kappa_2, \kappa_3, \dots\}$ of S . Denoting by γ also the measured lamination supported by the curve

γ with unitary weight, we have for every $h \in \mathcal{T}_S$ that

$$\begin{aligned}\varpi_h(e_l^\lambda, e_l^\gamma) &= 2 \left(d\ell_\gamma \wedge d\tau_\gamma + \sum_i d\ell_{\kappa_i} \wedge d\tau_{\kappa_i} \right) (e_l^\gamma, e_l^\lambda) = \\ &= d\ell_\gamma(e_l^\lambda) = dL_\gamma(e_l^\lambda) = \sum_{n=1}^N \frac{d}{dt}|_0 L_\gamma(E_l^{t\lambda_n}(h)).\end{aligned}$$

Kerckhoff in [29] proved that on a closed surface S if γ and ν are laminations with a closed curve as support then for every h in the Teichmüller space of S the following holds:

$$(4) \quad \frac{d}{dt}|_0 L_\gamma(E_l^{t\nu}(h)) = \int_\gamma \cos \theta_{(\gamma, \nu)}(h) d\nu$$

where $\theta_{(\gamma, \nu)}(h)$ denotes the angle measured counterclockwise from γ to ν in the h -realization. In the proof in [29] of Equation (4) the fact that ν was a closed curve was actually irrelevant. Thus, in our context, the same argument shows that for any h in \mathcal{T}_S and $\nu \in \mathcal{ML}_S$

$$\frac{d}{dt}|_0 L_\gamma(E_l^{t\nu}(h)) = \int_\gamma \cos \theta_{(\gamma, \nu)}(h) d\nu.$$

Therefore,

$$\varpi(e_l^\lambda, e_l^\gamma) = \sum_{n=1}^N \int_\gamma \cos \theta_{(\gamma, \lambda)} d\lambda_n.$$

If a function $H: \mathcal{T}_S(\mathbf{b}) \rightarrow \mathbb{R}$ verifies

$$dH(e_l^\gamma) = \sum_{n=1}^N \int_\gamma \cos \theta_{(\gamma, \lambda)} d\lambda_n$$

then, since the space of simple weighted closed curves is dense in \mathcal{ML}_S^c , by an approximation argument we get that for every $\nu \in \mathcal{ML}_S^c$

$$(5) \quad dH(e_l^\nu) = \varpi(e_l^\lambda, e_l^\nu).$$

Thus, by definition, H is Hamiltonian of the field e_l^λ .

If $\lambda_1, \dots, \lambda_N$ have compact support, with the same argument one gets that $H = -\sum_n L_{\lambda_n}$ is a suitable Hamiltonian. In the following sections we will show that it is always possible to construct a Hamiltonian $-\mathbb{L}_\lambda$ of e_l^λ for every $\lambda \in \mathcal{ML}_S^\#$.

2.3. Circuital laminations. If λ_1 and λ_2 are measured laminations with empty transverse intersection, their sum $\lambda_1 \oplus \lambda_2$ is defined by putting $\text{supp}(\lambda_1 \oplus \lambda_2) = \text{supp}(\lambda_1) \cup \text{supp}(\lambda_2)$ and $\text{meas}_{\lambda_1 \oplus \lambda_2} = \text{meas}_{\lambda_1} + \text{meas}_{\lambda_2}$. By example, if $\lambda = (\delta, \omega)$ is a weighted curve and $\omega = \omega_1 + \omega_2$ then λ is the sum of $\lambda_1 = (\delta, \omega_1)$ and $\lambda_2 = (\delta, \omega_2)$. It is immediate to see that

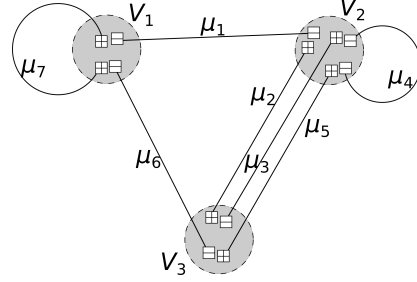
$$(6) \quad e_l^{\lambda_1 \oplus \lambda_2} = e_l^{\lambda_1} + e_l^{\lambda_2}.$$

Definition 2.1. We say that a I -uple $\mu = (\mu_1, \dots, \mu_I)$ of laminations is a circuital lamination if for every $i = 1, \dots, I$

- μ_i are ω -weighted single spiralling leaves;

- μ_1, \dots, μ_I are oriented so that for every if μ_{i-1} ends spiralling near $D_i \in \{\partial_1, \dots, \partial_n\}$ then μ_i starts spiralling near D_i , providing $\mu_0 = \mu_I$;
- the spiralling sense of μ_{i-1} near D_i is opposite to the one of μ_i near D_i .

A graphic interpretation of such definition can be obtained constructing a multigraph as follows. Take n vertices V_1, \dots, V_n , representing respectively the boundary components $\partial_1, \dots, \partial_n$ of S . For every leaf μ_i spiralling from ∂_m to ∂_k draw an edge from V_m to V_k , marking each endpoint with \boxminus if the leaf spirals in negative sense, with \boxplus otherwise. The I -uple (μ_1, \dots, μ_I) is circuital if it corresponds to a cycle that every time it passes from an edge to another one switches the sign of the endpoint.



Remark 2.1. If (μ_1, \dots, μ_I) is a circuital lamination, then, looking at the corresponding multigraph, for every boundary component ∂_k of S

$$\sum_{i=1}^I (\partial_k, \mu_i) = \omega \cdot \left(\#\{\boxplus \text{ in } V_k\} - \#\{\boxminus \text{ in } V_k\} \right) = 0.$$

Therefore, $\mathcal{ML}_S^\#$ contains all the circuital laminations.

Proposition 2.1. *For every $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathcal{ML}_S^\#$ there exist circuital laminations $\mu^{(1)}, \dots, \mu^{(J)}$ such that*

$$(7) \quad e_l^\lambda = e_l^{\lambda^{(0)}} + \sum_{j=1}^J e_l^{\mu^{(j)}}$$

where $\lambda^{(0)}$ is the N -uple of the compact parts of $\lambda_1, \dots, \lambda_n$.

Proof. If $\lambda = (\lambda_1^{(0)}, \dots, \lambda_N^{(0)})$ there is nothing to prove. Otherwise, consider the multigraph G associated with $\text{spir}(\lambda) = (\lambda_1 \setminus \lambda_1^{(0)}, \dots, \lambda_N \setminus \lambda_N^{(0)})$. We start by looking for a circuital lamination $\mu^{(1)} = (\mu_1^{(1)}, \dots, \mu_{I_1}^{(1)})$ contained in $\text{spir}(\lambda)$; this is equivalent to find a cycle in the graph G alternating the signs of the endpoints of the edges (notice that such cycle is allowed to pass on an edge more than one time).

Since $\lambda \in \mathcal{ML}_S^\#$, a vertex V of G contains a \boxplus symbol if and only if V also contains a \boxminus symbol, since the condition $\sum m(\partial, \lambda_n) = 0$ implies that near ∂ laminations can not all spiral in the same sense.

Let us start from a vertex D^0 reached by an endpoint \boxminus of an edge $\hat{\mu}_1$ and denote by D^1 the vertex (maybe coincident with D^0) of the other endpoint of $\hat{\mu}_1$. If such endpoint has the \boxminus symbol, there must be a \boxplus symbol in D^1 , endpoint of an edge $\hat{\mu}_2$; vice versa, if such endpoint has the \boxplus symbol, there must be a \boxminus symbol in D^1 , endpoint of an edge $\hat{\mu}_2$. Denote by D_2 the vertex

of the other endpoint of $\hat{\mu}_2$ and reiterate to find D^3 and $\hat{\mu}_3$, always switching endpoint symbols. Following such procedure, we get a switching path on G (in the sense that consecutive edges have opposite endpoint symbols). If we can find K such that there is $H < K$ and the subpath from D^H to D^K is a switching cycle, then we have finished. We claim that if we visit a vertex D^k for the third time then either we have already found such K (and it is less than k) or there is $H < k$ such that the path from D^H to D^k is a switching cycle (so k is the K we were looking for). Suppose we visit a D^k for the third time without having found a switching cycle before. Then

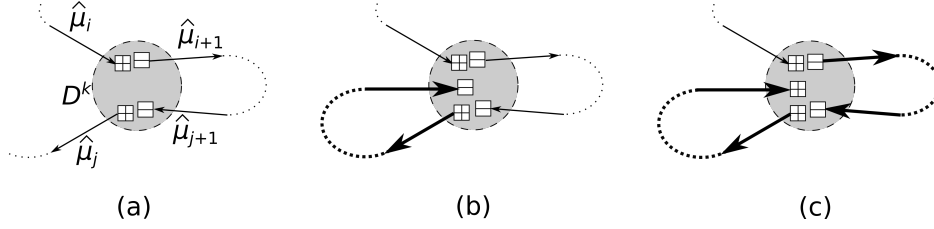


FIGURE 5

the configuration of the previous two visits must be the one in Figure 5 (a), up to exchanging \boxplus and \boxminus . The third time the path enters D^k , it can add either a \boxminus symbol, as in Figure 5 (b), or a \boxplus symbol, as in Figure 5 (c). In both case, a switching cycle can be individuated, as enlightened in the pictures.

So there exists a switching cycle

$$(\mu_1^{(1)}, \mu_2^{(1)}, \dots, \mu_{I_1}^{(1)}) = (\hat{\mu}_H, \hat{\mu}_{H+1}, \dots, \hat{\mu}_{K-1})$$

in G , generating a circuitual lamination $\mu^{(1)}$ contained in λ .

We want to endow $\mu^{(1)}$ with a weight $\omega^{(1)}$ so that if $\Lambda = (\Lambda_1, \dots, \Lambda_N)$ is the N -uple of laminations such that

$$e_t^\lambda = e_t^{\lambda^{(0)}} + e_t^{\mu^{(1)}} + e_t^\Lambda$$

then $\mu^{(1)}$ has at least one leaf not contained in the support of Λ . For every spiralling leaf δ of λ , denote by ω_δ its weight. Define

$$\omega^{(1)} = \min \left\{ \frac{\omega_\delta}{\#\{i \in \{1, \dots, I_1\} \mid \mu_i^{(1)} = \delta\}} \mid \delta \text{ is a leaf of } \lambda \right\}.$$

In this way, the leaf of λ where such minimum is achieved does not appear in the support of Λ .

If Λ is the N -uple of void laminations, we have finished. See Figure 6 as example, where the cycle in (b) spans the triple of laminations in (a). Otherwise, notice that again $\Lambda \in \mathcal{ML}_S^\#$ (it depends on the fact that $\mu^{(1)}$ lies in $\mathcal{ML}_S^\#$; see Remark 2.1). Moreover Λ has less leaves than λ . By a simple inductive argument we get circuitual sublaminations $\mu^{(1)}, \dots, \mu^{(J)}$, with $J \in \mathbb{N}$, such that (7) holds. \square



coordinates such that a preimage of ∂ coincides with the imaginary ray and a lift \tilde{l} of l is $1 + i\mathbb{R}_{>0}$. Here we are supposing that l spirals around ∂ in, say, positive sense.

Set $b = \ell(\partial)$ and let $\gamma: z \mapsto e^b z$ denote the holonomy transformation corresponding to ∂ . The union of the lifts of m with an ideal endpoint in 0 is γ -invariant. Among them, there exists a unique \tilde{m} such that $\tilde{l} \cap \gamma^k(\tilde{m})$ is non-empty for every $k \geq 0$ and $\tilde{l} \cap \gamma^k(\tilde{m})$ is empty for every $k < 0$. For every $k \geq 0$ let \tilde{p}_k be the intersection between \tilde{l} and $\gamma^k(\tilde{m})$ and p_k the projection of \tilde{p}_k on S .

A simple geometrical analysis shows that p_0 satisfies the stated properties. \square

Remark 2.2. Let us consider the points \tilde{p}_k chosen as in the proof of the previous lemma. They belong to \tilde{l} , so $\Re \tilde{p}_k = 1$ for every k . The geodesic m spirals around ∂ in the opposite sense of l , so an ideal endpoint of \tilde{m} must be 0. The other endpoint of \tilde{m} is $\cos^{-2} \phi$, where $\phi = \arg \tilde{p}_0$. This implies that $\gamma^k(\tilde{m})$ has ideal endpoints 0 and $e^{bk} \cos^{-2} \phi$. From this, for every $k \geq 0$ we can compute the imaginary part of the points $\tilde{p}_k = l \cap \gamma^k(\tilde{m})$:

$$\tilde{p}_k = \tilde{l} \cap \gamma^k(\tilde{m}) = 1 + i\sqrt{e^{bk} \cos^{-2} \phi - 1}.$$

Lemma 2.3. Fix $\mathbf{b} \in (\mathbb{R}_{>0})^n$. For every boundary component ∂ of S there exists $\varepsilon(\partial) > 0$ such that for every $h \in \mathcal{T}_S(\mathbf{b})$ every simple complete geodesic that enters the $\varepsilon(\partial)$ -collar $\mathcal{N}(\partial)$ of ∂ exits no more.

Proof. Choose $h \in \mathcal{T}_S(\mathbf{b})$ and set $b = \ell(\partial)$. On the universal cover $\mathcal{H} \subset \mathbb{H}^2$ take coordinates such that the imaginary ray projects on a boundary component ∂ . Let $\gamma: z \mapsto e^b z$ be the corresponding holonomy transformation. If the endpoints $z < z'$ of a complete geodesic $\tilde{\sigma}$ in \mathcal{H} are such that $z' > e^b z$, then $z < \gamma(z) < z' < \gamma(z')$, so $\gamma(\tilde{\sigma})$ meets $\tilde{\sigma}$. Therefore, if a geodesic $\sigma \subset S$ is simple and not spiralling around ∂ , any lift $\tilde{\sigma}$ must have endpoints $z < z'$ such that $z' < e^b z$. A standard computation shows that σ does not enter a $\varepsilon(\partial)$ -collar of ∂ , where

$$\varepsilon(\partial) = \frac{1}{\tanh(b/2)}.$$

\square

For every boundary component ∂ of S , we will denote by $\mathcal{N}(\partial)$ the $\varepsilon(\partial)$ -collar of ∂ and we will call the union \mathcal{N} of such collars *spiralization neighbourhood*.

Remark 2.3. If $k \geq 1$ then p_k lies in $\mathcal{N}(\partial)$. In fact, a point x of l lies in $\mathcal{N}(\partial)$ if and only if the preimage of x on \tilde{l} has imaginary part greater than $\tan \varphi(\partial) = \sinh(b/2)$ (see Lemma 2.3). For $k \geq 1$ we have

$$\Im \tilde{p}_k \geq \Im \tilde{p}_1 = \sqrt{e^b \cos^{-2} \phi - 1} \geq \sqrt{e^b - 1} \geq \sinh(b/2).$$

It may be possible that p_0 does not lie in $\mathcal{N}(\partial)$. That is the reason why the definition of L will involve p_1 and not p_0 .

Remark 2.4. If $k \geq 1$, the distance between p_k and ∂ is computed by

$$\tanh d(p_k, \partial) = \tanh d(\tilde{p}_k, \tilde{\partial}) = \cos \arg \tilde{p}_k = \frac{\Re \tilde{p}_k}{|\tilde{p}_k|} = e^{-bk/2} \cos \phi.$$

Now let us come back to the circuital lamination λ with leaves $\lambda_1, \dots, \lambda_I$. Let $p^{[i]} = p_1^{[i]}$ be the point p_1 near D_i chosen as in the proof of Lemma 2.2 when $l = \lambda_{i-1}$ and $m = \lambda_i$, providing $\lambda_0 = \lambda_I$. Now we can define a map $L = L_\lambda: \mathcal{T}_S(\mathbf{b}) \rightarrow \mathbb{R}$, that will turn out in Subsection 2.5 to be the opposite of a Hamiltonian of e_l^λ .

Definition 2.2. Take an ω -weighted circuital lamination λ , and consider the points $p^{[i]}$ introduced above. Let ρ be the union of the geodesic arcs in λ_i with endpoints $p^{[i]}$ and $p^{[i+1]}$ on $i = 1, \dots, I$. For every $h \in \mathcal{T}_S(\mathbf{b})$, set

$$L(h) = \omega \left\{ \ell_h(\rho) + 2 \log \prod_{i=1}^I \cosh d_h(p^{[i]}, D_i) \right\}.$$

We notice that L depends on the circuital decomposition of λ .

Remark 2.5. Consider the loops ρ_k made by the truncations of the leaves λ_i at the points $p_k^{[i]}$ relative to D_i (defined as in Lemma 2.2), so that $\rho_1 = \rho$. Notice that $\rho_{k+1} \setminus \rho_k$ is a union of M loops, each isotopic to a certain D_i . Moreover, such loops tend to some components of ∂S , as k goes to infinity. Setting

$$B_h = \sum_{i=1}^I \ell_h(D_i),$$

it turns out that the map

$$h \mapsto \omega(\ell_h(\rho_k) + 2 \log \prod_{i=1}^I \cosh d_h(p_k^{[i]}, D_i) - k B_h)$$

is independent on k . See [36] for details. Therefore, the map $L_k: \mathcal{T}_S \rightarrow \mathbb{R}$ defined by

$$L_k(h) = \omega \ell_h(\rho_k) + 2\omega \log \prod_{i=1}^I \cosh d_h(p_k^{[i]}, D_i)$$

differs from $L = L_1$ by $(k-1)B_h$, a constant depending only on the h -lengths of the boundary components.

2.5. The first order variation of L . The goal of this Subsection is to prove the following proposition:

Proposition 2.4. Take an ω -weighted circuit of laminations $\lambda \in \mathcal{ML}_S^\#$ and consider the map $L = L_\lambda: \mathcal{T}_S(\mathbf{b}) \rightarrow \mathbb{R}$ given by Definition 2.2. For every non-peripheral and non-trivial simple close curve γ on S and for every $h \in \mathcal{T}_S(\mathbf{b})$ the equation

$$(8) \quad \frac{d}{dt} L(E_l^{t\gamma}(h)) = \sum_{i=1}^I \int \cos \theta_{(\lambda_i, \gamma)}(t) d\gamma \otimes d\lambda_i$$

holds, where $\theta_{(\lambda_i, \gamma)}(t)$ is the angle measured counterclockwise from the support of λ_n to γ , in the $E_l^{t\gamma}(h)$ -realization of γ and λ_i .

Notice that we are slightly abusing the notation, denoting by γ also the measured lamination supported by the curve γ with unitary weight. This

proposition will be true more in general, replacing γ with a measured lamination ν with compact support, as shown at the end of the Subsection. Since

$$L(h) = \omega \left\{ \ell_h(\rho) + 2 \log \prod_{i=1}^I \cosh d_h(p^{[i]}, D_i) \right\},$$

we will first compute the derivative in $t = 0$ of $\omega \ell_{E_t^\gamma(h)}(\rho)$, which will turn out to be

$$\omega \frac{d}{dt} \Big|_{t=0} \ell_{E_t^\gamma(h)}(\rho) = \sum_{i=1}^I \int \cos \theta_{(\lambda_i, \gamma)}(0) d\gamma \otimes d\lambda_i + \omega \sum_{i=1}^I \mathcal{R}_i(0)$$

where \mathcal{R}_i are terms due to the presence of the vertices $p^{[i]}$ in ρ . After that, setting $F(d) = 2 \log \cosh d$, we will show that

$$(9) \quad \mathcal{R}_i(0) + \frac{d}{dt} \Big|_{t=0} F(d_{E_t^\gamma(h)}(p^{[i]}, D_i)) = 0$$

thus proving Equation (8).

Let us start to compute the derivative of $\ell_{E_t^\gamma(h)}(\rho)$. Notice that the loop ρ is piecewise geodesic and has exactly I vertices, which are $p^{[i]}$ for $i = 1, \dots, I$. If $\iota(\gamma, \lambda_i) = 0$ for every i then $\ell(\rho)$ is constant. Otherwise, γ meets at

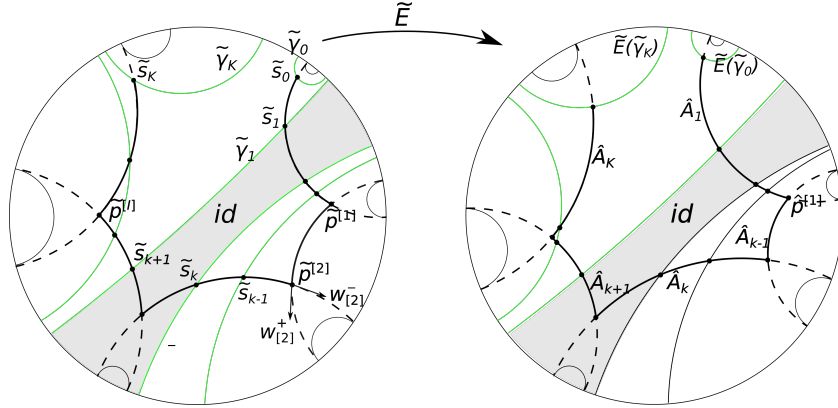


FIGURE 7. Determination of $\hat{\rho}$ and \hat{A}_j (here $I = 4$)

least one λ_i . Notice that $\gamma \cap \rho = \gamma \cap \bigcup \lambda_i$, since $p^{[i]}$ lies in the spiralization neighbourhood for every i (see Lemma 2.3 and Remark 2.3). Choosing an orientation of ρ , enumerate consecutively its intersections with γ as s_0, s_1, \dots, s_{K-1} . Pick a preimage \tilde{s}_0 of s_0 on the universal cover \mathcal{H} of S . If $r: [0, 1] \rightarrow S$ is a parametrization of the loop ρ such that $r(0) = r(1) = s_0$, take the lift $\tilde{r}: [0, 1] \rightarrow \mathcal{H}$ with $\tilde{r}(0) = \tilde{s}_0$. Put $\tilde{s}_K = \tilde{r}(1)$ and \tilde{s}_k the preimage of s_k along \tilde{r} for $k < K$. The preimages of γ determine the strata of the lifting \tilde{E} of $E_l^{t\gamma}$. In particular, denote by $\tilde{\gamma}_k$ the preimage of γ passing through \tilde{s}_k , for $k = 0, \dots, K$.

The path \tilde{r} is piecewise geodesic, with vertices $\tilde{p}^{[i]}$. The images of the lifts of the components of ∂S through \tilde{E} , together with $\tilde{E}(\tilde{\gamma}_0)$ and $\tilde{E}(\tilde{\gamma}_K)$, determine the piecewise geodesic arc $\hat{\rho}$ (which does not coincide with $\tilde{E}(\tilde{r})$)

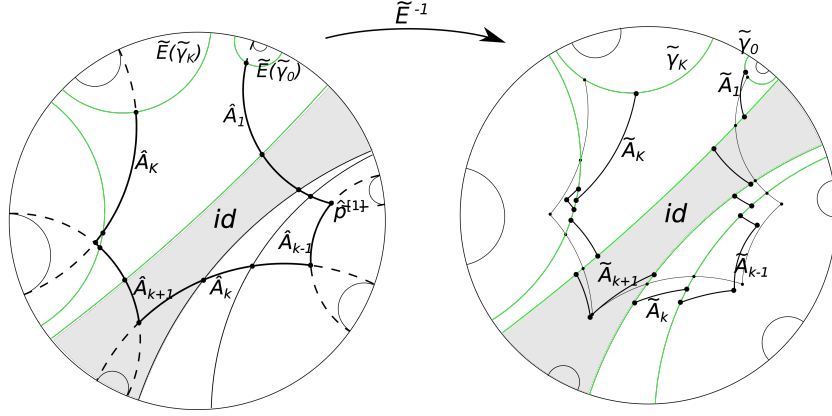


FIGURE 8. Determination of A_j (here $I = 4$)

whose length is equal to $\ell_{E_l^{t\gamma}(h)}(\rho)$. The arc $\hat{\rho}$ is divided in K piecewise geodesic subarcs $\hat{A}_1, \dots, \hat{A}_K$ by its intersections with $\bigcup \tilde{E}(\tilde{\gamma}_k)$; such subarcs are enumerated following the orientation of $\hat{\rho}$. The preimage A_k under \tilde{E} of \hat{A}_k is a piecewise geodesic arc with endpoints $x_k \in \tilde{\gamma}_{k-1}$ and $y_k \in \tilde{\gamma}_k$ with the same length as \hat{A}_k . Notice that $x_1 = \tilde{s}_0$ and $y_K = \tilde{s}_K$. This leads to

$$\ell_{E_l^{t\gamma}(h)}(\rho) = \sum_{k=1}^K \ell_h(A_k(t)).$$

For $k = 1, \dots, K$ denote with v_k the unitary vector tangent to \tilde{r} at $\tilde{s}_k = x_{k-1}(0) = y_k(0)$, by θ_k the angle in \tilde{s}_k measured counterclockwise from \tilde{r} to $\tilde{\gamma}_k$ and by u_k the unitary tangent vector to $\tilde{\gamma}_k$ at \tilde{s}_k such that $\pi - \theta_k$ is the angle between v_k and u_k , as in Figure 9. Notice that

$$\sum_{i=1}^I \int \cos \theta_{(\lambda_i, \gamma)} d\gamma \otimes d\lambda_i = \omega \sum_{k=1}^K \cos \theta_k.$$

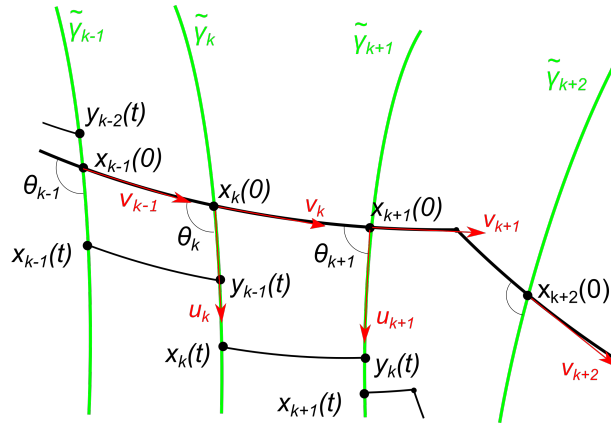


FIGURE 9

Lemma 2.5. For $k = 1, \dots, K - 1$, the following identity holds:

$$(10) \quad \dot{x}_{k+1}(0) = \dot{y}_k(0) + u_k.$$

Proof. Denote by $d_k(t)$ the signed distance between $y_k(0) = x_{k+1}(0)$ and $y_k(t)$ on $\tilde{\gamma}_k$ oriented as u_k . Then

$$\begin{aligned} y_k(t) &= y_k(0) \cosh d_k(t) + u_k \sinh d_k(t) \\ x_{k+1}(t) &= x_{k+1}(0) \cosh(d_k(t) + t) + u_k \sinh(d_k(t) + t). \end{aligned}$$

Therefore,

$$\begin{aligned} \dot{y}_k(0) &= u_k \dot{d}_k(0) \\ \dot{x}_{k+1}(0) &= u_k (\dot{d}_k(0) + 1) \end{aligned}$$

leading to (10). \square

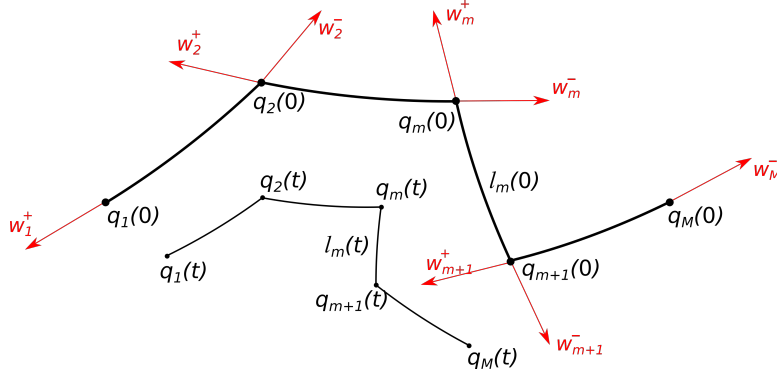


FIGURE 10

Lemma 2.6. Consider the hyperboloid model of \mathbb{H}^2 in $\mathbb{R}^{2,1} = (\mathbb{R}^3, \langle *, * \rangle)$ (where $\langle x, y \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2$), namely

$$\mathbb{H}^2 \cong \{x \in \mathbb{R}^{2,1} : \langle x, x \rangle = -1, x_0 > 0\}.$$

Given an integer $M \geq 2$ and a C^1 map $q: [0, 1] \rightarrow (\mathbb{H}^2)^M$, let $C(t)$ be the oriented open polygonal chain in \mathbb{H}^2 of vertices $q_1(t), \dots, q_M(t)$. Denote by w_m^- and $-w_m^+$ respectively the left and right unitary tangent vector to $C(0)$ at $q_m(0)$, for $m = 2, \dots, M - 1$. Define analogously $-w_1^+$ and w_M^- , as in Figure 10. Then

$$\frac{d}{dt}\bigg|_0 \ell(C(t)) = \langle \dot{q}_1(0), w_1^+ \rangle + \langle \dot{q}_M(0), w_M^- \rangle + \sum_{m=2}^{M-1} \langle \dot{q}_m(0), w_m^- + w_m^+ \rangle.$$

Proof. Set $l_m(t) = d(q_m(t), q_{m+1}(t))$. It suffices to prove that

$$(11) \quad \dot{l}_m(0) = \langle \dot{q}_m(0), w_m^+ \rangle + \langle \dot{q}_{m+1}(0), w_{m+1}^- \rangle$$

for $m = 1, \dots, M - 1$. Since $\cosh l_m(t) = -\langle q_m(t), q_{m+1}(t) \rangle$, differentiating at $t = 0$ we get

$$(12) \quad \dot{l}_m(0) \sinh l_m(0) = -\langle \dot{q}_m(0), q_{m+1}(0) \rangle - \langle q_m(0), \dot{q}_{m+1}(0) \rangle.$$

Since

$$\begin{aligned} q_{m+1}(0) &= q_m(0) \cosh l_j(0) - w_m^+ \sinh l_m(0) \\ q_m(0) &= q_{m+1}(0) \cosh l_j(0) - w_{m+1}^- \sinh l_m(0), \end{aligned}$$

equation (12) becomes

$$\dot{l}_m(0) \sinh l_m(0) = \langle \dot{q}_m(0), w_m^+ \rangle \sinh l_m(0) + \langle \dot{q}_{m+1}(0), w_{m+1}^- \rangle \sinh l_m(0),$$

which gives (11). \square

We are able now to prove the following result.

Proposition 2.7.

$$\frac{d}{dt}|_0 \sum_{k=1}^K \ell_h(A_k(t)) = \sum_{k=1}^K \cos \theta_k + \sum_{i=1}^I \mathcal{R}_i(0)$$

where $\mathcal{R}_1, \dots, \mathcal{R}_I$ are terms related to the I vertices of ρ (explicitly computed in the proof, see Equation (13)).

Proof. Each A_k is a piecewise geodesic arc, with endpoints x_k and y_k . Applying Lemma 2.6 to every A_k , we get

$$\begin{aligned} \sum_{k=1}^K \frac{d}{dt}|_0 \ell_h(A_k(t)) &= \sum_{k=1}^K \left(\langle \dot{x}_k(0), -v_{k-1} \rangle + \langle \dot{y}_k(0), v_k \rangle \right) + \\ &+ \sum_{i=1}^I \langle \dot{p}^{[i]}(0), w_-^{[i]} + w_+^{[i]} \rangle, \end{aligned}$$

where $\pm w_{\pm}^{[i]}$ denote the unitary vectors tangent to \tilde{r} at $\tilde{p}^{[i]}$ and the vectors v_k where defined before Lemma 2.5.

Let us put

$$\begin{aligned} \mathcal{S} &= \sum_{k=1}^K \left(\langle \dot{x}_k(0), -v_{k-1} \rangle + \langle \dot{y}_k(0), v_k \rangle \right) \\ (13) \quad \mathcal{R}_i(0) &= \langle \dot{p}^{[i]}(0), w_-^{[i]} + w_+^{[i]} \rangle \end{aligned}$$

Using (10), we have that

$$\begin{aligned} \mathcal{S} &= \langle \dot{x}_1(0), -v_0 \rangle + \sum_{k=2}^K \langle \dot{x}_k(0), -v_{k-1} \rangle + \sum_{k=1}^{K-1} \langle \dot{y}_k(0), v_k \rangle + \langle \dot{y}_K(0), v_K \rangle = \\ &= -\langle \dot{x}_1(0), v_0 \rangle + \sum_{k=1}^{K-1} \langle \dot{x}_{k+1}(0) - \dot{y}_k(0), -v_k \rangle + \langle \dot{y}_K(0), v_K \rangle = \\ &= -\langle \dot{x}_1(0), v_0 \rangle + \sum_{k=1}^{K-1} \langle u_k, -v_k \rangle + \langle \dot{y}_K(0), v_K \rangle. \end{aligned}$$

Since s_0 is a point where ρ is smooth and $\tilde{s}_0 = x_1(0)$ and $\tilde{s}_m = y_K(0)$ are preimages of s_0 , there exists a covering transformation T such that

$T\dot{x}_1(0) = \dot{y}_K(0) + u_K$ and $Tv_0 = v_K$. Now

$$\begin{aligned}\mathcal{S} &= -\langle T\dot{x}_1(0), Tv_0 \rangle - \sum_{k=1}^{K-1} \langle u_k, v_k \rangle + \langle \dot{y}_K(0), v_K \rangle = \\ &= -\langle \dot{y}_K(0) + u_K, v_K \rangle - \sum_{k=1}^{K-1} \langle u_k, v_k \rangle + \langle \dot{y}_K(0), v_K \rangle = \\ &= -\sum_{k=1}^K \langle u_k, v_k \rangle = \sum_{k=1}^K \cos \theta_k.\end{aligned}$$

□

Now we have to show that Equation (9) holds.

Let us first recall some known facts on the hyperboloid model of \mathbb{H}^2 , keeping the notation of the proof of Proposition 2.7. For every geodesic γ in \mathbb{H}^2 there is a space-like vector n_γ such that

$$\gamma = \{\underline{x} \in \mathbb{H}^2 \mid \langle \underline{x}, n_\gamma \rangle = 0\}.$$

The boundary at infinity of \mathbb{H}^2 is identified with

$$\partial_\infty \mathbb{H}^2 = \{\underline{x} \in \mathbb{R}^{2,1} \mid \langle \underline{x}, \underline{x} \rangle = 0\} / \underline{x} \sim a\underline{x}, a \in \mathbb{R}^*$$

and its elements will be written within square brackets. See also [9].

There is a notion of cross product in $\mathbb{R}^{2,1}$, analogous to the Euclidean environment: if dV denotes the volume form in $\mathbb{R}^{2,1}$, the cross product between $\underline{x} \in \mathbb{R}^{2,1}$ and $\underline{y} \in \mathbb{R}^{2,1}$ is the vector $\underline{x} \boxtimes \underline{y} \in \mathbb{R}^{2,1}$ such that for every $\underline{z} \in \mathbb{R}^{2,1}$

$$\langle \underline{x} \boxtimes \underline{y}, \underline{z} \rangle = dV(\underline{x}, \underline{y}, \underline{z}).$$

The following hold:

$$\begin{aligned}\langle \underline{x}, \underline{y} \boxtimes \underline{z} \rangle &= \langle \underline{z}, \underline{x} \boxtimes \underline{y} \rangle \\ (\underline{x} \boxtimes \underline{y}) \boxtimes \underline{z} &= \langle \underline{y}, \underline{z} \rangle \underline{x} - \langle \underline{x}, \underline{z} \rangle \underline{y} \\ \langle \underline{x} \boxtimes \underline{y}, \underline{x} \boxtimes \underline{y} \rangle &= \langle \underline{x}, \underline{y} \rangle^2 - \langle \underline{x}, \underline{x} \rangle \langle \underline{y}, \underline{y} \rangle\end{aligned}$$

for every $\underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^{2,1}$.

Now fix i and denote by ∂ the component of ∂S whose spiralization neighbourhood contains $p^{[i]}$. If $\tilde{\partial}$ is the lift of ∂ closer to $\tilde{p} = \tilde{p}^{[i]}$, denote by $[z^+]$ and $[z^-]$ the ideal endpoints of $\tilde{\partial}$, so that $w^\pm = w_\pm^{[i]}$ is pointing towards $[z^\pm]$.

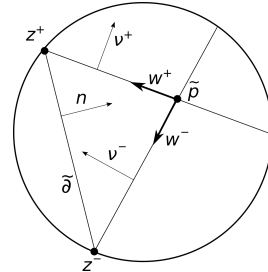
The unitary vector

$$n = \frac{z^- \boxtimes z^+}{\|z^- \boxtimes z^+\|_{2,1}}$$

is the normal unitary vector of $\tilde{\partial}$ pointing towards

\tilde{p} . Up to precomposing by a proper isometry, we can suppose that $[z^+]$ and $[z^-]$ are kept fixed by \tilde{E} , thus $\tilde{E}(n) = n$. If $d = d(p, \partial) = d(\tilde{p}, \tilde{\partial})$, then $\sinh d = \langle \tilde{p}, n \rangle$. Therefore

$$\dot{d} = \frac{\langle \dot{\tilde{p}}, n \rangle}{\cosh d}$$



and

$$\frac{d}{dt}F(d) = 2 \frac{\sinh d}{\cosh^2 d} \langle \dot{\tilde{p}}, n \rangle,$$

where we have set $F(d) = 2 \log \cosh d$. Now Equation (9) becomes

$$\left\langle \dot{\tilde{p}}, w^+ + w^- + 2 \frac{\sinh d}{\cosh^2 d} n \right\rangle = 0.$$

The following proposition will prove such equation computing w^\pm in terms of \tilde{p} and n .

Proposition 2.8.

$$\langle \dot{\tilde{p}}, w^+ + w^- \rangle = -2 \frac{\sinh d}{\cosh^2 d} \langle \dot{\tilde{p}}, n \rangle.$$

Proof. The vector w^\pm can be written as $\tilde{p} \boxtimes \nu^\pm$, where ν^\pm is the unitary vector tangent to \mathbb{H}^2 and normal to w^\pm (i.e. to $\lambda_{[i-1]}/\lambda_{[i]}$) oriented in the proper way; namely,

$$\nu^\pm = -\frac{z^\pm \boxtimes \tilde{p}}{\|z^\pm \boxtimes \tilde{p}\|_{2,1}}.$$

Thus,

$$w^\pm = -\tilde{p} \boxtimes \frac{z^\pm \boxtimes \tilde{p}}{\|z^\pm \boxtimes \tilde{p}\|_{2,1}} = -\frac{-\langle \tilde{p}, \tilde{p} \rangle z^\pm + \langle z^\pm, \tilde{p} \rangle \tilde{p}}{\langle z^\pm, \tilde{p} \rangle} = -\frac{z^\pm + \langle z^\pm, \tilde{p} \rangle \tilde{p}}{\langle z^\pm, \tilde{p} \rangle}.$$

We claim that

$$(14) \quad z^\pm = \tilde{p} - (\sinh d)n \pm \tilde{p} \boxtimes n.$$

First, we have to see that the right hand side of (14) is a null vector; let us compute the square norm of $\tilde{p} \boxtimes n$:

$$\langle \tilde{p} \boxtimes n, \tilde{p} \boxtimes n \rangle = \langle \tilde{p}, n \rangle^2 - \langle \tilde{p}, \tilde{p} \rangle \langle n, n \rangle = \sinh^2 d + 1 = \cosh^2 d$$

Now

$$\begin{aligned} & \langle \tilde{p} - (\sinh d)n \pm \tilde{p} \boxtimes n, \tilde{p} - (\sinh d)n \pm \tilde{p} \boxtimes n \rangle = \\ & = \langle \tilde{p}, \tilde{p} \rangle - (\sinh d) \langle \tilde{p}, n \rangle - (\sinh d) \langle n, \tilde{p} \rangle + (\sinh^2 d) \langle n, n \rangle + \langle \tilde{p} \boxtimes n, \tilde{p} \boxtimes n \rangle = \\ & = -1 - \sinh^2 d - \sinh^2 d + \sinh^2 d + \cosh^2 d = 0. \end{aligned}$$

On the other hand, we have to check that $\tilde{p} - (\sinh d)n \pm \tilde{p} \boxtimes n$ are the ideal endpoints of $\tilde{\delta}$ (or equivalently $\langle \tilde{p} - (\sinh d)n \pm \tilde{p} \boxtimes n, n \rangle = 0$) such that

$$\left(\langle \tilde{p} - (\sinh d)n - \tilde{p} \boxtimes n, \tilde{p} - (\sinh d)n + \tilde{p} \boxtimes n, n \rangle \right)$$

forms a negative basis of $\mathbb{R}^{2,1}$. Now

$$\langle \tilde{p} - (\sinh d)n \pm \tilde{p} \boxtimes n, n \rangle = \langle \tilde{p}, n \rangle - (\sinh d) \langle n, n \rangle = 0$$

and

$$\begin{aligned} & \langle \tilde{p} - (\sinh d)n - \tilde{p} \boxtimes n, (\tilde{p} - (\sinh d)n + \tilde{p} \boxtimes n) \boxtimes n \rangle = \\ & = \langle \tilde{p} - (\sinh d)n - \tilde{p} \boxtimes n, \tilde{p} \boxtimes n + \tilde{p} \rangle = -1 - \cosh^2 d < 0. \end{aligned}$$

Thus, we can compute

$$\langle z^\pm, \tilde{p} \rangle = \langle \tilde{p} - (\sinh d)n \pm \tilde{p} \boxtimes n, \tilde{p} \rangle = -\cosh^2 d$$

and

$$w^\pm = -\frac{z^\pm + \langle z^\pm, \tilde{p} \rangle \tilde{p}}{\langle z^\pm, \underline{p} \rangle} = -\frac{(\sinh^2 d) \tilde{p} + (\sinh d) n \mp \tilde{p} \boxtimes n}{\cosh^2 d}.$$

Now

$$\langle \dot{\tilde{p}}, w^+ + w^- \rangle = \left\langle \dot{\tilde{p}}, -\frac{(2 \sinh^2 d) \tilde{p} + (2 \sinh d) n}{\cosh^2 d} \right\rangle = -2 \frac{\sinh d}{\cosh^2 d} \langle \dot{\tilde{p}}, n \rangle.$$

□

Finally, let us consider the first order variation of $t \mapsto L(E_l^{t\nu}(h))$ in the general case, when $\nu \in \mathcal{ML}_S^c$.

Proposition 2.9. *Consider a circuital lamination $\lambda \in \mathcal{ML}_S^\#$. For every $h \in \mathcal{T}_S(\mathbf{b})$ and $\nu \in \mathcal{ML}_S^c$ the following formula holds:*

$$(15) \quad \frac{d}{dt} L_\lambda(E_l^{t\nu}(h)) = \sum_{i=1}^I \int \cos \theta_{(\lambda_i, \nu)}(t) d\nu \otimes d\lambda_i.$$

Proof. The space of weighted curves on S is dense in \mathcal{ML}_S^c (see [34]), so take a sequence (γ_j) of weighted curves converging to ν . With the notation

$$\text{Cos}(\lambda, \mu)(t) = \int \cos \theta_{(\lambda, \mu)}(t) d\mu \otimes d\lambda$$

used in [31], we have seen that

$$\frac{d}{dt} L_\lambda(E_l^{t\gamma_j}(h)) = \sum_{i=1}^I \text{Cos}(\lambda_i, \gamma_j)(t).$$

Clearly $L_\lambda(E_l^{t\gamma_j}(h))|_{t=0} = L_\lambda(E_l^{t\nu}(h))|_{t=0}$ for every j , so if we prove that $\sum_i \text{Cos}(\lambda_i, \gamma_j)$ tends uniformly to $\sum_i \text{Cos}(\lambda_i, \nu)$ then $L_\lambda(E_l^{t\gamma_j}(h))$ tends to $L_\lambda(E_l^{t\nu}(h))$ and (15) holds. Kerckhoff showed in [31] itself that $\text{Cos}(\delta, \gamma_j)$ tends uniformly to $\text{Cos}(\delta, \nu)$ for every δ closed curve in S , but his argument still works if δ is a spiralling leaf of a lamination on S , so we can conclude. □

2.6. The map $\mathbb{L}_\lambda: \mathcal{T}_S(\mathbf{b}) \rightarrow \mathbb{R}$. In order to extend the definition of L_λ to any $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathcal{ML}_S^\#$, consider the decomposition

$$e_l^\lambda = e_l^{\lambda^{(0)}} + \sum_{j=1}^J e_l^{\mu^{(j)}}$$

by Proposition 2.1, where $\lambda^{(0)}$ is the compact part of λ and $\mu^{(j)}$ are circuital laminations. Now define

$$(16) \quad \mathbb{L}_\lambda = \sum_{j=1}^J L^{(j)}$$

where $L^{(0)} = \sum_n L_{\lambda_n^{(0)}}$ and $L^{(j)}$ is the length map of $\mu^{(j)}$ in Definition 2.2, for $j \neq 0$. Since Equation (8) holds for every $L^{(j)}$, we can deduce

$$\frac{d}{dt} \mathbb{L}_\lambda(E_l^{t\nu}(h)) = \sum_{n=1}^N \int \cos \theta_{(\lambda_n, \nu)} d\nu \otimes d\lambda$$

for every $\nu \in \mathcal{ML}_S^c$ and $h \in \mathcal{T}_S(\mathbf{b})$. In particular, $-\mathbb{L}_\lambda$ is a Hamiltonian of e_t^λ (see Equation (5)).

3. PROPERTIES OF \mathbb{L}_λ

3.1. \mathbb{L}_λ is proper. Now we are going to show that the map \mathbb{L}_λ is proper under the hypothesis that $\lambda = (\lambda_1, \dots, \lambda_N)$ fills up S , which means that every non-trivial non-peripheral simple closed curve on S meets $\bigcup \text{supp}(\lambda_n)$. Set

$$\mathcal{FML}_S^\# = \{\lambda \in \mathcal{ML}_S^\# \mid \lambda \text{ fills up } S\} \cup \{(\emptyset, \emptyset)\}.$$

As explained in section 1.3 any spiralling geodesic γ of a measured geodesic lamination can be replaced by a geodesic arc γ^R orthogonal to the boundary. For each $\nu \in \mathcal{ML}_S$ denote by ν^R the set of geodesic arcs obtained by ν replacing each spiralling geodesic γ of ν with γ^R and set $\lambda^R = ((\lambda_1)^R, \dots, (\lambda_N)^R)$. Notice that if $\lambda \in \mathcal{FML}_S^\# \setminus \{(\emptyset, \emptyset)\}$ then λ^R still fills up S .

Lemma 3.1. *Consider two disjoint geodesics ∂ and ∂' in \mathbb{H}^2 , a geodesic γ going from an endpoint of ∂ to an endpoint of ∂' , the geodesic arc γ^R with endpoints on ∂ and ∂' normal to ∂ and ∂' , two positive real numbers $\epsilon, \epsilon' \leq \ell(\gamma^R)/2$, the ϵ -collars N of ∂ and the ϵ' -collar N' of ∂' . Then*

$$\ell(\gamma \setminus (N \cup N')) \geq \ell(\gamma^R \setminus (N \cup N')) = \ell(\gamma^R) - \epsilon - \epsilon'.$$

□

Such lemma is quite easy to prove; see [36] for details.

Proposition 3.2. *If $\lambda \in \mathcal{FML}_S^\# \setminus \{(\emptyset, \emptyset)\}$ then the map $\mathbb{L}_\lambda: \mathcal{T}_S(\mathbf{b}) \rightarrow \mathbb{R}$ is proper.*

Proof. Choose a pant decomposition of S with curves $\kappa_1, \dots, \kappa_{3(g-1)+n}$, $\partial_1, \dots, \partial_n$ and consider the related coordinates

$$(l_1, \dots, l_{3(g-1)+n}, \tau_1, \dots, \tau_{3(g-1)+n})$$

on $\mathcal{T}_S(\mathbf{b})$, where l_i is the length of κ_i and τ_i is the twist factor on κ_i . Choose also for every κ_i two dual curves κ_i^* and κ_i^{**} whose lengths can reconstruct τ_i (as explained in [23]; see Figure 11).

We have seen at the beginning of this subsection that if $\lambda \in \mathcal{FML}_S^\#$ then λ^R fills up S ; this implies that every simple closed non-trivial curve in S is isotopic to a curve on $G = D \cup \bigcup \text{supp}((\lambda_n)^R)$, where $D = \bigcup \partial_j$.

We claim that

$$\begin{aligned} \mathbb{L}_G: \mathcal{T}_S(\mathbf{b}) &\longrightarrow \mathbb{R} \\ h &\mapsto \sum_{j=1}^n b_j + \sum_{n=1}^N \ell_h((\lambda_n)^R) \end{aligned}$$

is a proper map. Pick a divergent sequence $\{h_k\}$ in $\mathcal{T}_S(\mathbf{b})$; then the sequence

$$\{(l_1, \dots, l_{3(g-1)+n}, \tau_1, \dots, \tau_{3(g-1)+n})(h_k)\}$$

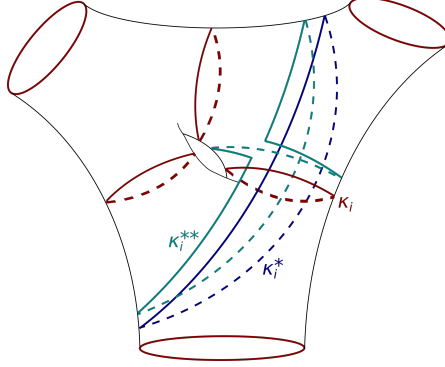


FIGURE 11

is divergent in $\mathbb{R}^{6(g-1)+2n}$. This implies that

$$S_k = \sum_{i=1}^{3(g-1)+n} [\ell_{h_k}([\kappa_i]) + \ell_{h_k}([\kappa_i^*]) + \ell_{h_k}([\kappa_i^{**})] \xrightarrow{n \rightarrow \infty} +\infty,$$

where for any closed curve κ and hyperbolic metric h we denote by $\ell_h([\kappa])$ the h -length of the geodesic h -realization of κ .

Each κ_i (and κ_i^* and κ_i^{**}) is isotopic to many (not necessarily simple) curves in G , but for every i the number

$$m_i = \min \left\{ \max_{p \in G} \left\{ \#(\pi^{-1}(p) \cap ([0, 1] \times \{0\})) \right\} \mid \pi: [0, 1] \times [0, 1] \rightarrow S \text{ isotopy} \right. \\ \left. \text{between } \pi(*, 0) = \kappa_i \text{ and } \pi(*, 1) \text{ closed curve in } G \right\},$$

which denotes a sort of minimum of the degrees of the isotopies between κ_i and any curve in G , does not depend on the metric. The same holds for m_i^* and m_i^{**} (the analogous numbers for κ_i^* and κ_i^{**} respectively). If m_0 is the maximum among all m_i 's, m_i^* 's and m_i^{**} 's, then

$$S_k \leq 3m_0(3g - 3 + n)\mathbb{L}_G(h_k).$$

Therefore, $\{\mathbb{L}_G(h_k)\}$ is going to infinity as $\{h_k\}$ is diverging.

Since $\mathbb{L}_G(h_k) = \sum b_i + \sum \ell_{h_k}((\lambda_n)^R)$ is diverging, two possibilities occur:

- a compact sublamination γ^R of λ^R has divergent length; but since $\gamma^R = \gamma$, also $\mathbb{L}_\lambda(h_k)$ is diverging;
- no compact sublamination of λ^R has divergent length; then an arc γ^R in λ^R (replacement of a spiralling leaf γ of λ between ∂ and ∂') has divergent length. Also $\ell_{h_k}(\gamma \setminus (\mathcal{N}(\partial) \cup \mathcal{N}(\partial')))$ diverges, by Lemma 3.1, where $\mathcal{N}(\partial)$ is the $\varepsilon(\partial)$ -collar introduced in Subsection 2.4. From the definition,

$$\mathbb{L}_\lambda(h_k) > \omega \ell_{h_k}(\gamma - \mathcal{N}(\partial) - \mathcal{N}(\partial')) > \omega \left(\ell_{h_k}(\gamma^R) - \varepsilon(\partial) - \varepsilon(\partial') \right),$$

implying that $\mathbb{L}_\lambda(h_k)$ is diverging.

□

3.2. The second order variation of \mathbb{L}_λ . If $\lambda \in \mathcal{FML}_S^\# \setminus \{(\emptyset, \emptyset)\}$, the map $\mathbb{L}_\lambda: \mathcal{T}_S(\mathbf{b}) \rightarrow \mathbb{R}_{\geq 0}$ is strictly convex along left earthquakes, which means that $t \mapsto \mathbb{L}(E_l^{t\nu}(h))$ is strictly convex for every ν in $\mathcal{ML}_S^\varepsilon$ and every $h \in \mathcal{T}_S(\mathbf{b})$. Kerckhoff proved it in [29] for $\nu \in \mathcal{ML}_S^\varepsilon$, but his argument still applies to spiralling laminations: he worked on the universal cover of the surface \tilde{S} , where the key-point was that any right (respectively left) earthquake induces a homeomorphism on $\partial_\infty \tilde{S}$ that moves clockwise (respectively counterclockwise), which still is true in our context.

Remark 3.1. As in [29], properness and strict convexity of $\mathbb{L}: \mathcal{T}_S(\mathbf{b}) \rightarrow \mathbb{R}$ assure that \mathbb{L} admits exactly one point of minimum h_0 .

The goal of this subsection is to show that the Hessian of \mathbb{L}_λ is positive definite on a critical point $h_0 \in \mathcal{T}_S(\mathbf{b})$ of \mathbb{L}_λ . If $\lambda_1, \dots, \lambda_N$ have compact discrete support, then the result is already known through explicit formulas (see [42], [19]), which however involve quantities that are not meaningful in our setting. Let us consider $\nu \in \mathcal{ML}_S^\varepsilon$. We already know from Subsections 2.5 and 2.6 that

$$\frac{d}{dt} \mathbb{L}_\lambda(E_l^{t\nu}(h_0)) = \sum_{n=1}^N \int \cos \theta_{(\lambda_n, \nu)}(t) d\nu \otimes d\lambda_n$$

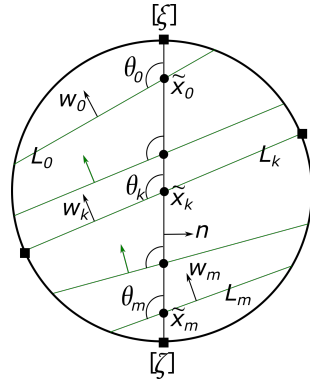
holds, where $\theta_{(\lambda_n, \nu)}(t)$ is the angle measured counterclockwise from the support of λ_n to ν , in the $E_l^{t\nu}(h_0)$ -realization of ν and λ_n .

The compact part of ν is approximated by closed weighted curves, so let us consider first a unitary closed curve γ . If δ is a weighted spiralling leaf of λ , we will first compute

$$\frac{d}{dt}|_0 \int \cos \theta_{(\delta, \gamma)}(t) d\gamma \otimes d\delta = \sum_{i=1}^m \frac{d}{dt}|_0 \cos \theta_i,$$

where, enumerating consecutively along δ the points x_1, \dots, x_m in $\zeta \cap \gamma$, θ_i is the angle measured counterclockwise from δ to γ at x_i . Then we will deduce an estimate which guarantees that even passing at the limit of closed curves the second derivative stays positive.

Let us transfer the problem on the universal covering $\mathcal{H} \subset \mathbb{H}^2$ of S in the hyperboloid model of \mathbb{H}^2 . Fix a lift $\tilde{\delta}$ of δ ; denote by $\tilde{x}_1, \dots, \tilde{x}_m$ the preimages of x_1, \dots, x_m on $\tilde{\delta}$ and by L_1, \dots, L_m the liftings of γ passing respectively through $\tilde{x}_1, \dots, \tilde{x}_m$. Denote by $[\xi]$ and $[\zeta]$ the ideal endpoints of $\tilde{\delta}$ so that $\tilde{x}_1, \dots, \tilde{x}_m$ are enumerated from $[\xi]$ to $[\zeta]$ and $\xi_0 = \zeta_0 = 1$, if we write vectors \underline{x} in $\mathbb{R}^{2,1}$ as $\underline{x} = (x_0, x_1, x_2)$. We can choose coordinates such that $\langle \xi, \zeta \rangle = -1$. Fix $k \in \{1, \dots, m\}$ and consider the lift \tilde{E}^t of $E_l^{t\gamma}$ which fixes the gap whose boundary contains L_k and L_{k-1} (if $k = 1$ take the earthquake that fixes the gap adjacent with L_1 whose ideal boundary contains $[\xi]$). Choose unitary vectors w_1, \dots, w_m normal respectively to L_1, \dots, L_m so



that $\cos \theta_i = \langle w_k, n \rangle$ for every i . Now, since we are in the hyperboloid model of \mathbb{H}^2 , let us identify $\mathbb{R}^{2,1}$ with the Lie algebra $\mathfrak{so}(2,1)$. Now

$$\begin{aligned}\xi(t) &= \tilde{E}^t(\xi) = \exp(-tw_1) \cdots \exp(-tw_{k-1})\xi, \\ \zeta(t) &= \tilde{E}^t(\zeta) = \exp(+tw_k) \cdots \exp(+tw_m)\zeta, \\ n(t) &= \frac{\xi(t) \boxtimes \zeta(t)}{\|\xi(t) \boxtimes \zeta(t)\|_{2,1}} = \frac{\xi(t) \boxtimes \zeta(t)}{-\langle \xi(t), \zeta(t) \rangle}\end{aligned}$$

so

$$(17) \quad \dot{\xi}(0) = -\sum_{i=1}^{k-1} w_i \boxtimes \xi,$$

$$(18) \quad \dot{\zeta}(0) = \sum_{i=k}^m w_i \boxtimes \zeta.$$

Since

$$\frac{d}{dt}|_0 \cos \theta_k(t) = \frac{d}{dt}|_0 \langle w_k, n(t) \rangle = \langle w_k, \dot{n}(0) \rangle,$$

let us compute $\dot{n}(0)$. In general,

$$\begin{aligned}\dot{n}(0) &= \frac{\dot{\xi}(0) \boxtimes \zeta + \xi \boxtimes \dot{\zeta}(0)}{-\langle \xi \boxtimes \zeta \rangle} + \frac{\xi \boxtimes \zeta}{\langle \xi \boxtimes \zeta \rangle^2} \frac{d}{dt}|_0 \langle \xi(t), \zeta(t) \rangle = \\ &= \dot{\xi}(0) \boxtimes \zeta + \xi \boxtimes \dot{\zeta}(0) + n \cdot \frac{d}{dt}|_0 \langle \xi(t), \zeta(t) \rangle.\end{aligned}$$

Setting $z = \dot{\xi}(0) \boxtimes \zeta + \xi \boxtimes \dot{\zeta}(0)$, we deduce that there is $\beta \in \mathbb{R}$ such that $\dot{n}(0) = z + \beta n$. So from

$$0 = \langle \dot{n}(0), n \rangle = \langle z, n \rangle + \beta \langle n, n \rangle = \langle z, n \rangle + \beta$$

we get

$$\dot{n}(0) = z - \langle z, n \rangle n.$$

Writing $\dot{\xi}$ for $\dot{\xi}(0)$ and $\dot{\zeta}$ for $\dot{\zeta}(0)$, setting for every i

$$w_i = a_i \xi + b_i \zeta + c_i n,$$

(notice that $a_i > 0 > b_i$) and using (17), (18), we compute z as

$$\begin{aligned}z &= \dot{\xi} \boxtimes \zeta + \xi \boxtimes \dot{\zeta} = -\sum_{i=1}^{k-1} (w_i \boxtimes \xi) \boxtimes \zeta + \sum_{i=k}^m \xi \boxtimes (w_i \boxtimes \zeta) = \\ &= \sum_{i=1}^{k-1} b_i \zeta + \sum_{i=k}^m a_i \xi + \sum_{i=1}^m c_i n.\end{aligned}$$

Now

$$\frac{d}{dt}|_0 \cos \theta_k(t) = \langle w_k, \dot{n}(0) \rangle = \langle w_k, z \rangle - \langle z, n \rangle \langle w_k, n \rangle.$$

The three products take values

$$\begin{aligned}
\langle w_k, z \rangle &= \left\langle a_k \xi + b_k \zeta + c_k n, \sum_{i=1}^{k-1} b_i \zeta + \sum_{i=k}^m a_i \xi + \sum_{i=1}^m c_i n \right\rangle = \\
&= - \sum_{i=1}^{k-1} a_k b_i - \sum_{i=k}^m a_i b_k + \sum_{i=1}^m c_i c_k \\
\langle z, n \rangle &= \left\langle \sum_{i=1}^{k-1} b_i \zeta + \sum_{i=k}^m a_i \xi + \sum_{i=1}^m c_i n, n \right\rangle = \sum_{i=1}^m c_i \\
\langle w_k, n \rangle &= \langle a_k \xi + b_k \zeta + c_k n, n \rangle = c_k
\end{aligned}$$

so

$$\frac{d}{dt}|_0 \cos \theta_k(t) = - \sum_{i=1}^{k-1} a_k b_i - \sum_{i=k}^m a_i b_k.$$

The sum over k gives

$$\sum_{k=1}^m \frac{d}{dt}|_0 \cos \theta_k(t) = - \sum_{k=1}^m \sum_{i=1}^{k-1} a_k b_i - \sum_{k=1}^m \sum_{i=k}^m a_i b_k = - \sum_{k=1}^m a_k b_k - 2 \sum_{i < k} a_i b_k.$$

Notice that $c_k = \langle w_k, n \rangle = \cos \theta_k$ and

$$1 = \langle w_k, w_k \rangle = -2a_k b_k + c_k^2,$$

which implies $-a_k b_k = (\sin^2 \theta_k)/2$. The terms $r_{ik} = -a_i b_k > 0$ have the property that $r_{ik} r_{ki} = (\sin^2 \theta_i \sin^2 \theta_k)/4$; moreover,

$$\cosh d(L_i, L_k) = \langle w_i, w_k \rangle = -a_i b_k - a_k b_i + c_i c_k = r_{ik} + r_{ki} + \cos \theta_i \cos \theta_k.$$

Since $d(L_i, L_k)$ is bounded by the maximal length of a curve contained in $\bigcup \text{supp}(\lambda_n) \setminus \mathcal{N}$, there is $M_0 > 0$ such that

$$r_{ik} + r_{ki} = \cosh d(L_i, L_k) - \cos \theta_i \cos \theta_k \leq \cosh M_0 + 1.$$

Now

$$r_{ik} = \frac{r_{ik} r_{ki}}{r_{ki}} \geq \frac{r_{ik} r_{ki}}{r_{ik} + r_{ki}} \geq \frac{\sin^2 \theta_i \sin^2 \theta_k}{4(\cosh M_0 + 1)}.$$

We finally get

$$\sum_{k=1}^m \frac{d}{dt}|_0 \cos \theta_k(t) \geq \frac{1}{2} \left(\sum_{k=1}^m \sin^2 \theta_k + \sum_{i < k} \frac{\sin^2 \theta_i \sin^2 \theta_k}{2(\cosh M_0 + 1)} \right).$$

This holds for a spiralling leaf δ in λ . Considering all the leaves of λ , there is $M_1 > 0$ such that we obtain

$$\frac{d^2}{dt^2} \mathbb{L}(E_t^\gamma(h_0)) \geq M_1 \sum_{n=1}^N \int_{\lambda_n} \int_{\lambda_n} \sin^2 \theta_{(\lambda_n, \gamma)}(x) \sin^2 \theta_{(\lambda_n, \gamma)}(y) d\gamma(x) d\gamma(y)$$

or equivalently

$$\text{Hess } \mathbb{L}(e_l^\gamma, e_l^\gamma)|_{h_0} \geq M_1 \sum_{n=1}^N \iint \sin^2 \theta_{(\lambda_n, \gamma)}(x) \sin^2 \theta_{(\lambda_n, \gamma)}(y) d\gamma(x) d\gamma(y).$$

Now let us consider a generic $\nu \in \mathcal{ML}_S^c$. It is the limit of weighted closed curves γ_k . As for the first order variation of \mathbb{L}_λ , with an approximation argument we get that

$$\text{Hess } \mathbb{L}(e_l^\nu, e_l^\nu)|_{h_0} \geq M_1 \sum_{n=1}^N \iint \sin^2 \theta_{(\lambda_n, \nu)}(x) \sin^2 \theta_{(\lambda_n, \nu)}(y) d\nu(x) d\nu(y).$$

Therefore, $\text{Hess}_{h_0} \mathbb{L}_\lambda$ is definite positive.

Remark 3.2. When $N = 2$, consider $\lambda = (\lambda_1, \lambda_2) \in \mathcal{FML}_S^\# \setminus \{(\emptyset, \emptyset)\}$ and let h_0 be the unique critical point of \mathbb{L}_λ . Since $-\mathbb{L}_\lambda$ is the symplectic gradient of e_l^λ with respect to ϖ , as shown in Section 2, saying that $\text{Hess}_{h_0} \mathbb{L}_\lambda > 0$ is equivalent to state that $e_l^{\lambda_1}$ and $-e_l^{\lambda_2} = e_r^{\lambda_2}$ meet transversely (only) in h_0 . If $\lambda \in \mathcal{ML}_S^\# \setminus \mathcal{FML}_S^\#$, with $N \geq 2$, the map \mathbb{L}_λ does not have any critical points. Kerckhoff proved it for $n = 0$ and $N = 2$ in [31] (Theorem 2.1 II), but the same argument works for any $n \geq 0$ and $N \geq 2$, since the key point was that Equation (15) holds.

4. THE TANGENT SPACE

In this section we extend a result achieved in [18], Appendix B., using the enlightened properties of the Hamiltonian \mathbb{L}_λ of the vector field $e_l^\lambda + e_l^\mu$. Let $\mathbf{M}^3 = (\mathbb{R}^{2,1}, \langle *, * \rangle)$ be the 3-dimensional Minkowski space and consider \mathbb{H}^2 as the set of unitary future-pointing vectors of \mathbf{M}^3 , preserved by the action of $SO_0(2, 1)$. In this section, we identify \mathcal{T}_S with the space of cocompact Fuchsian representations $h: \pi_1(S) \rightarrow SO_0(2, 1)$ up to conjugacy. An affine deformation of h is a representation

$$\rho = h + \tau: \pi_1(S) \rightarrow \mathbb{R}^3 \rtimes SO_0(2, 1) \subset \text{Isom}(\mathbf{M}^3) = \mathbb{R}^3 \rtimes O(2, 1)$$

with $\tau(\gamma) \in \mathbb{R}^3$ for every $\gamma \in \pi_1(S)$. The space $T\mathcal{T}_S$ is identified with

$$\{\rho = h + \tau: \pi_1(S) \rightarrow \mathbb{R}^3 \rtimes SO_0(2, 1) : h \in \mathcal{T}_S\} / \text{conj.},$$

the space of affine deformations of Fuchsian representations.

In [3], Barbot proved that for every $\rho \in \mathcal{R}(\mathbf{b})$ there are two maximal disjoint convex non-empty domains $\Omega^\pm(\rho) \subset \mathbf{M}^3$ such that

- $\Omega^+(\rho)$ (respectively $\Omega^-(\rho)$) is complete in the future (respectively in the past);
- the action of $\rho(\pi_1(S))$ on $\Omega^\pm(\rho)$ is free and properly discontinuous;
- $\rho(\pi_1(S)) \backslash \Omega^\pm(\rho) \simeq S \times \mathbb{R}$ is a maximal Cauchy-hyperbolic spacetime.

Being $\Omega^\pm(\rho)$ regular domains, they are associated with two measured laminations λ_\pm , considered as dual to the singular loci of $\Omega^\pm(\rho)$. See [8] for details. Denote by $\Psi: T\mathcal{T}_S \rightarrow \mathcal{ML}_S^2$ the arising map. As in [18], if $\Psi(h, \tau) = (\lambda_+, \lambda_-)$ then

$$(19) \quad \tau = e_r^{\lambda_+}(h) = e_l^{\lambda_-}(h).$$

Now we can state the following proposition, which immediately leads to Theorem B

Proposition 4.1. *Fix $\mathbf{b} \in (\mathbb{R}_{>0})^n$ and set $V = T\mathcal{T}_S|_{\mathcal{T}_S(\mathbf{b})}$. The restriction*

$$\Psi_0^{\mathbf{b}} = \Psi|_{V \setminus V_0}: V \setminus V_0 \rightarrow \mathcal{ML}_S \times \mathcal{ML}_S,$$

where V_0 denotes the zero section of V , is bijective onto $\mathcal{FML}_S^\# \setminus \{(\emptyset, \emptyset)\}$.

Proof. Put

$$\Psi^{\mathbf{b}} = \Psi|_V : V \rightarrow \mathcal{ML}_S \times \mathcal{ML}_S.$$

Clearly $\Psi^{\mathbf{b}}(V_0) = \{(\emptyset, \emptyset)\}$.

In the first place, let us show that $\Psi^{\mathbf{b}}(V) \subseteq \mathcal{ML}_S^\#$. From (19), if $\Psi^{\mathbf{b}}(h, \tau) = (\lambda_+, \lambda_-)$ then $e_i^{\lambda_+} + e_i^{\lambda_-} = 0$. Using (2) we get

$$0 = \frac{d}{dt}\bigg|_0 \left(\ell_{E_i^{\lambda_+}(h)}(\partial_i) + \ell_{E_i^{\lambda_-}(h)}(\partial_i) \right) = -m(\partial_i, \lambda_+) - m(\partial_i, \lambda_-)$$

for $i = 1, \dots, n$.

In order to see that $\Psi_0^{\mathbf{b}}$ is injective onto $\mathcal{FML}_S^\# \setminus \{(\emptyset, \emptyset)\}$, we need to prove that for every $(\lambda_+, \lambda_-) \in \mathcal{ML}_S \times \mathcal{ML}_S$ there is a 1:1 correspondence between $(\Psi^{\mathbf{b}})^{-1}(\lambda_+, \lambda_-)$ and $e_r^{\lambda_+} \cap e_l^{\lambda_-}$. If $(\lambda_+, \lambda_-) = \Psi^{\mathbf{b}}(h, \tau)$ then $e_r^{\lambda_+}$ and $e_l^{\lambda_-}$ meet over h by construction. Conversely, if there exists $h \in \mathcal{T}_S$ such that $e_r^{\lambda_+}(h) = e_l^{\lambda_-}(h)$, then $(\lambda_+, \lambda_-) = \Psi^{\mathbf{b}}(h, e_r^{\lambda_+}(h))$.

Now, we showed in Section 3 for $(\lambda_+, \lambda_-) \neq (\emptyset, \emptyset)$ that $e_l^{\lambda_+} + e_l^{\lambda_-}$ has exactly one critical point if (λ_+, λ_-) is a filling couple, zero critical points otherwise (Remark 3.2). Therefore, $\Psi_0^{\mathbf{b}}$ is injective with image $\mathcal{FML}_S^\# \setminus \{(\emptyset, \emptyset)\}$. \square

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